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PERFORMANCE CRITERIA FOR LINEAR CONSTANT-COEFFICIENT SYSTEMS WITH DETERMINISTIC INPUTS

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NOTICES

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FOREWORD

This report represents one phase of an effort directed at the use of performance criteria as elements in flight control system optimization studies. The research reported was sponsored by the Flight Control Laboratory of the Aeronautical Systems Division under Project No. 8219. It was conducted at Systems Technology, Inc. under Contract No. AF 33(616)-7841. The ASD project engineers were Mr. R. O. Anderson and Lt. L. Schwartz of the Flight Control Laboratory. The principal investigators were Messrs. D. T. McRuer and Dunstan Graham. The principal contributors to this report are listed as authors. Messrs. D. T. McRuer and Dunstan Graham, Systems Technology, Inc. technical directors, planned the general approach and contributed many details. J. Wolkovitch, STI project engineer, wrote the report and originated the material of Chapters IV and V. Ray Magdaleno originated Appendices A and B, and John McDonnell produced the graphs and detailed relationships of Chapter II.

The authors wish to express their thanks to Mr. V. J. Kovacevich for his diligent work in generating the ITE^2 and IT^2E^2 general and standard forms, and to Lt. L. Schwartz for his thorough check of the ITAE derivations and for many valuable comments. Acknowledgment is gratefully made to Mr. J. Taira, and to Mr. H. R. Pass who contributed many of the calculations, and to Messrs. W. E. Ellis and R. N. Nye for their careful work in preparing the final illustrations for this report.

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ABSTRACT

Performance measures and associated criteria for linear constant coefficient systems forced by deterministic inputs are investigated, with particular reference to flight control systems. It is shown that the application of performance measures is facilitated by substituting for the actual flight control system an equivalent low-order linearized system having similar dynamic characteristics. A critical survey of current performance measures is given, and new methods for the analytic evaluation of some indicial error measures are presented.

Numerous criteria are examined with regard to their validity, selectivity, and ease of application. Normalized presentations are used so that practical limitations on the time scale of the response (e.g., due to power/inertia restrictions) can be taken into account separately. It is concluded that minimum ITAE (integrated time moment of absolute error) and minimum ITE^2 (integrated time moment of error-squared) have particular merit. The ITAE criterion yields smooth indicial responses having little overshoot, but its analytic description is complicated. Of the criteria—minimum IE^2 (integrated error-squared), minimum ITE^2, IT^2E^2, IT^3E^2 (integrated first-, second-, and third time moments of error-squared)—IE^2 has simple analytic forms, but selects poor indicial responses; IT^3E^2 responses are as good as those selected by ITAE, but IT^3E^2 (and also IT^2E^2) analytic expressions are too complicated for general use. ITE^2 selects moderately smooth and well-damped responses (less good than ITAE), but possesses tractable analytic forms. Therefore, ITE^2 is recommended for analytic investigations, whereas ITAE is preferred for optimization using analog computers. Some other criteria also appear promising for use in conjunction with digital computers, but require further investigation to determine their validity and selectivity.

PUBLICATION REVIEW

This report has been reviewed and is approved.

FOR THE COMMANDER:

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NOMENCLATURE

$A(t)$	$= ae^{-\gamma t}$ (in Appendix B)
A_m	$= 2ke^{-\sigma T_1}/\pi$ (in Appendix B)
$[A]$	Matrix of coefficients (see Table I)
a	Coefficient of aperiodic term in error response
a_0	Argument of Gamma function ($= 1/2$)
a_1	Argument of Gamma function ($= 3/2$)
a_2	Argument of Gamma function ($= 5/2$)
a_i	Coefficient of s^i in characteristic equation, $\Delta(s) = 0$
a_i	Coefficient of s^{n-i} in denominator of $E(s)$
B	See Eq B-37 and B-38 (Appendix B)
$B(z,a_0)$	$= \Gamma(z)\Gamma(a_0)/\Gamma(z + a_0)$, the Beta function
b	$= ae^{-\delta t}$
b	Coefficient of s^2 in Eq 47
b_0	Normalized coefficient of s^2 in denominator of $C(s)/R(s)$
b_1	$= e^{-\delta T}$
$C(s)$	$= \mathcal{L}C(t)$
$C(t,\zeta)$	See Eq B-18 (Appendix B)
$C(t,m)$	$= 1/2\pi j\int_{\alpha-j\infty}^{\delta+j\infty} C(t,\zeta)d\zeta$
C_1, C_2, C_3, C_4	Paths of integration (see Fig. B-3)
c	$= \sqrt{1 - b^2}$ (in Appendix B)
c	Coefficient of s in Eq 47
c_0	Normalized coefficient of s in denominator of $C(s)/R(s)$

c_1	$= \sqrt{1 - b_1^2}$ (in Appendix B)
c_p	Peak overshoot
$c(t)$	Response
$c_e(t)$	Response envelope
D	Transfer function denominator
d_i	Coefficient of s^{n-i} in numerator of $E(s)$
$E(s)$	$= \mathcal{L}[e(t)]$
$E_A(s)$	$= \mathcal{L}\lvert e(t)\rvert$
E_{i-1}	i^{th} error coefficient
e	$= 2.718$
$e(t)$	Error response
$\lvert e(t)\rvert$	Absolute value of $e(t)$
$f_+(\omega)$	$= \mathcal{L}[h_+(x)]$
$f_-(\omega)$	$= \mathcal{L}[h_-(x)]$
$G(s)$	Transfer function
G_{rc}	$= C(s)/R(s) = G(s)/1 + G(s)$
G_{re}	$= E(s)/R(s) = 1/1 + G(s)$
$\lvert G_{re}\rvert$	Deviation ratio
g	$= \gamma - \alpha$
g_0	Argument of Beta function
H_0	See Eq B-44 (Appendix B)
H_1	See Eq B-83 (Appendix B)
h or $h(x)$	$= h_+ + h_-$, the transfer characteristic of the full-wave linear rectifier
h_+, h_-	That part of transfer characteristic for x positive or negative, respectively
IAE	Integral of absolute error, $\int_0^\infty \lvert e(t)\rvert dt$
IE	Integral of error, $\int_0^\infty e(t)dt$

IE^2	Integral of error-squared, $\int_0^\infty e(t)^2 dt$
ITAE	Integral of time-weighted absolute error, $\int_0^\infty t\lvert e(t)\rvert dt$
ITE	Integral of time-weighted error, $\int_0^\infty te(t)dt$
ITE^2	Integral of time-weighted error-squared, $\int_0^\infty t\, e(t)^2 dt$
IT^2E^2	Integral of time-squared times error-squared, $\int_0^\infty t^2 e(t)^2 dt$
IT^3E^2	Integral of time-cubed times error-squared, $\int_0^\infty t^3 e(t)^2 dt$
IT^nE	Integral of n^{th} time-moment of error, $\int_0^\infty t^n e(t)dt$
ITU	$\int_0^\infty tUdt$
IT^nU	$\int_0^\infty t^nUdt$
IU	Integral of a function of error not involving time explicitly, $\int_0^\infty Udt$
$I_n(z)$	Modified Bessel function of the first kind
$J_n(\eta)$	Bessel function
j	$= \sqrt{-1}$
K_1, K_2, K_3, K_4	See Eq B-19 (Appendix B)
k	Coefficient of oscillatory term in error response of third-order system (Appendix B)
k_1, k_2	See Eq B-61 (Appendix B)
$\mathcal{L}$	Laplace transform operator
$\mathcal{L}^{-1}$	Inverse Laplace transform operator
l_β	Rolling moment derivative with respect to sideslip
M_1, M_2	See Eq B-61 (Appendix B)
M_p	Peak magnification ratio
m	Order of "harmonics" in output signal

N Transfer function numerator

N Number of clockwise encirclements of -1 by $G(s)$

N_β Yawing moment derivative with respect to sideslip

N_η See Eq B-83 (Appendix B)

n Summation or infinite product index for Beta function

n System order

P Number of poles of $G(s)$ in right-half plane

P_i Coefficient of λ^i in the numerator of Eq 19

p Laplace transform variable if τ is independent variable

p_i Coefficient of λ^i in the numerator of Eq 23

Q_i Coefficient of λ^i in the denominator of Eq 19

q Exponent (Appendix B)

q_i Coefficient of λ^i in the denominator of Eq 23

q_i Coefficient of s^i in $C(s)/R(s)$ denominator

$R(s)$ $\mathcal{L}\, r(t)$

r Summation index

$r(t)$ Input

S_n $\int_0^\infty \left[\mathcal{L}^{-1} \frac{d_0 s^{n-1} + d_1 s^{n-2} + \cdots + d_{n-1}}{a_0 s^n + a_1 s^{n-1} + \cdots + a_n}\right]^2 dt$

s Laplace transform variable

s $= \sin^{-1} b$

s_1 $= \sin^{-1} b_1$

T A time constant

T_0 A time constant

T_1 $= [\ln(a/k)]/\gamma - \sigma$, time when ratio of envelopes equals one

T_e Elevator servo time constant

T_{eq} Equivalent time constant

T_1	Closed-loop time constant
t	Time, or normalized time
t_1	Time or normalized time
t_d	Delay time
t_p	Time to peak
t_R	Rise time
t_s	Settling time
U_m	See Eq B-44 (Appendix B)
u	$= \mathrm{Re}\ (\omega)$
U	An error function, not involving time explicitly
$V(t)$	$= ke^{-\alpha t}$
v	$= I_m(\omega)$
W or $W(t)$	See Eq B-41 (Appendix B)
$W_m(p)$	$\mathcal{L}\left[w_m(\tau)\right]$
$w_m(\tau)$	See Eq B-69 (Appendix B)
W_n	$\int_0^{\infty}\left[\mathcal{L}^{-1}\ \dfrac{d_0s^{n-1} + d_1s^{n-2} + \cdots + d_{n-1}}{a_0s^n + a_1s^{n-1} + \cdots + a_n}\right]^2 t\,dt$
I_n	$\int_0^{\infty}\left[\mathcal{L}^{-1}\ \dfrac{d_0s^{n-1} + d_1s^{n-2} + \cdots + d_{n-1}}{a_0s^n + a_1s^{n-1} + \cdots + a_n}\right]^2 t^2\,dt$
x	$= \mathrm{Re}\ (z)$
x or $x(t)$	Input to full-wave linear rectifier
Y	Open-loop transfer function
y	$= \mathrm{Im}\ (z)$
y or $y(t)$	Output of the full-wave linear rectifier

Z	Number of zeros of $1 + G(s)$ in right-half plane
Z or $Z(t)$	See Eq B-37 (Appendix B)
z	$= x + jy$ complex variable
α	Real part of complex roots of third-order system
$\alpha(s)$	Numerator of $G(s)$
β	Imaginary part of complex roots of third-order system
β	$= \omega_n\sqrt{1 - \zeta^2}$, or (Appendix A only) $\frac{x_1}{2\sqrt{1 - \zeta^2}}$
$\beta(s)$	Denominator of $G(s)$
$\Gamma(x)$	Gamma function
γ	Real root of third-order system transfer function denominator
γ	Laplace transform variable (in Chapter III)
Δ	Characteristic equation
Δ_1, Δ_2	See Eq B-61 (Appendix B)
Δ_3	$= \tan^{-1} \frac{2\beta\alpha}{\alpha^2 - \beta^2}$
Δ_4	$= \tan^{-1} \frac{\beta}{\alpha}$
δ	$= cV(t)$
δ	A time constant
δ_e	Elevator deflection
$\mathcal{E}_m$	$= 2 \quad m = 1, 2, \ldots$ Neumann factor
$\mathcal{E}_0$	$= 1$
ϵ	Real number greater than zero
ζ	$= c\bar{V}(t) = \xi + j\eta$, complex variable
ζ	Damping ratio
ζ_1	Closed-loop damping ratio
η	$= \mathrm{Im}\,(\zeta)$

θ	Angle of pitch
$\theta(t)$	$= \beta t + \psi - \frac{\pi}{2}$
κ	Open-loop gain
κ_c	Equalization gain
κ_m	Gain of servomotor plus amplifier combination
κ_1/κ	Gain margin
κ_ξ/κ	Generalized gain margin
λ	Open-loop time constant
λ	$= s + \sigma$
λ	$= \Omega_0 s$
μ	Magnitude of Laplace transform variable, s
μ_c	Generalized crossover frequency
ξ	$= \mathrm{Re}\ (\zeta)$
$-\xi$	Damping ratio of closed-loop roots
σ	Real part of complex variable, s
σ_1	A particular value of σ
τ	Time delay
τ	$= t - T_1$
ϕ	$= \psi + \beta T_1$ (phase angle)
ϕ_m	Phase margin
$\phi_{m\xi}$	Generalized phase margin
ψ	$= \tan^{-1} \frac{\beta}{\alpha} - \tan^{-1} \frac{\beta}{\gamma - \alpha}$ (phase angle)
ψ	$= \tan^{-1} \frac{\sqrt{1 - \zeta^2}}{\zeta}$
Ω_0	Normalized frequency
ω	Imaginary part of complex variable, $s = \sigma + j\omega$

ω	Laplace transform variable when x is the independent variable
ω_b	Bandwidth
ω_c	Crossover frequency
ω_1	Closed-loop root natural frequency
$\omega_n(\zeta)$	See Eq B-72 (Appendix B)
ω_n	Undamped natural frequency
ω_p	Peak magnification frequency
ω_i	Frequency of instability
$\measuredangle$	Angle
$\lvert\ \rvert$	Absolute value
$\lvert\ \rvert_{db}$	Magnitude in decibels
∂	Partial derivative
Σ	Summation
Π	Product

INTRODUCTION

Dynamic performance is only one of many factors to be considered in assessing the merit of a flight control system. Cost, weight, reliability, schedule, etc., must also be taken into account. The best choice for any given requirement can be found only by weighting each of these factors according to their relative importance. Cost, weight, etc., are measured directly and unequivocally in terms of dollars, pounds, etc., but at the present time assessment of dynamic performance depends heavily on intuition. This is not due to any shortage of performance measures; many have been proposed. The problem is which measure or combination of measures to choose, and, having made the choice, how to apply the resulting criteria to the system under consideration. It is to the solution of this problem that this report is directed.

The work reported here was performed under an Air Force contract directed at providing

1. A foundation for the specification of dynamic performance criteria for automatic flight control systems
2. The methods of analysis required to apply such criteria.

It was convenient to present the results of the study in two parts. The present report deals with performance criteria appropriate to deterministic inputs; random inputs and associated topics are discussed in a subsequent report.

Chapter I consists of a broad discussion of dynamic performance, and outlines exact and approximate calculation procedures. It is shown that although actual flight control systems are described by differential equations of high order, more tractable equivalent systems of low order can provide a convenient and sufficiently accurate basis for analytical optimization techniques. Qualities defining the merit of the dynamic performance of given systems are outlined, and their interpretation into numerical performance measures is discussed. Performance criteria are then defined in terms of optimal values of these measures, and the requirements of validity, selectivity, and ease of application that a good criterion should possess are formulated.

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Chapters II and III survey published measures of dynamic performance. Chapter II deals primarily with criteria not explicitly involving integral functions of error (e.g., phase margin, bandwidth, rise time, etc.), and presents extensive correlations of these criteria for second-order systems. Chapter III is concerned with indicial error measures; i.e., criteria directly measuring some integrated function of the error response to a step input. This class includes some of the most useful criteria, such as ITAE (integrated time moment of absolute error), IE^2 (integrated error-squared), etc. Calculation procedures for these criteria are presented; in particular, analytical expressions for ITAE and some other measures not previously expressed in analytical form are published for the first time. Chapter III concludes with a survey of the closed-loop and open-loop characteristics of those optimal systems which minimize particular indicial error measures.

Chapter IV derives general exact formulae expressing the effect of a pure time lag on system indicial error measures. By combining these formulae with the observation that the responses of high-order good and optimal systems closely approximate to the responses of lower-order systems with a time delay, a method of approximating to performance measures of high-order systems is devised. An example is given to show how this approximation simplifies optimization procedures.

Chapter V generalizes some of the formulae for indicial error measures to evaluate the corresponding measures for inputs other than steps. Relationships between open- and closed-loop forms of certain measures are also discussed.

Chapter VI discusses measures of sensitivity to parameter changes, and power requirements, and presents the conclusions reached from the present study on criteria for deterministic inputs.

CHAPTER I

CALCULATION OF DYNAMIC PERFORMANCE

The dynamic performance of a control system or element is assessed by considering the following factors:

1. Stability
2. Response to desired inputs
3. Response to unwanted inputs
4. Accuracy
5. Insensitivity to parameter changes
6. Power and/or energy demands.

The quantitative specification of dynamic performance consists of choosing measures of the above qualities (either singly or in combination), and setting desirable values or limits upon these measures. Defining a "performance measure" as a quantity characterizing dynamic performance, the term 'performance criterion" may be defined as a standard or reference value of a performance measure which provides a basis for a rule or test by which some aspect of dynamic performance is evaluated in forming a judgment of system quality.

For a performance criterion to be of use, it must be valid, selective, and readily applicable. Validity implies that the criterion is associated with desirable performance characteristics for the input environment of interest. The requirement of selectivity demands that the criterion should differentiate sharply between "good" systems and those which are merely "acceptable." For the criterion to be readily applicable, its expression in terms of system parameters should be compact, and convenient procedures for its evaluation should exist.

In principle, the process of designing a system to meet the specified performance consists of calculating the dynamic performance, applying the measure to the system under consideration, and then, if necessary, modifying the system so that the specified performance is attained or approached as closely as practicable. For convenience in implementing this step-by-step sequence, analytic procedures for the calculation of the performance meas should be available. However, many flight control systems are so complicated that their response to any given input can only be described by a differential

equation of high order, or by a large number of simultaneous differential equations of lower order. Physical realities thus tend to be obscured in a fog of mathematics. To avoid this situation, a number of simplifications can be introduced into the analytical representations of actual flight control systems. Foremost among these is the assumption of small perturbations and, hence, linearity. The further assumption is then commonly made that changes in vehicle configuration and environment occurring during the motion are small, so that the coefficients of the differential equations are effectively constant. The resulting linear constant-coefficient equations are still of high order. For purposes of calculation, a lower-order system which possesses (for a given input) approximately equivalent dominant mode dynamics can be constructed. This artifice is particularly valuable in the calculation of performance criteria. It will be demonstrated below that this more tractable equivalent system is relatively easy to deduce from the open-loop transfer function of the particular loop that is being studied. Later in this report it will also be shown that the result provides a sufficiently accurate basis for the calculation of performance criteria.

The general procedure by which equivalent systems are derived is most clearly illustrated by an example. Consider a pitch control system for the fighter airplane detailed in Appendix C. The open-loop transfer function for the pitch loop is

$$G(s) = \left\{ \frac{4.85\left(\frac{s}{1.372}+1\right)\left(\frac{s}{0.0098}+1\right)}{\left[\frac{s^2}{(0.0630)^2}+\frac{2(0.0714)}{0.0630}s+1\right]\left[\frac{s^2}{(4.27)^2}+\frac{2(0.493)}{4.27}s+1\right]} \right\}$$

Airplane Transfer Function

(1)

$$\times \left\{ \frac{K_E K_1\left(\frac{s}{2.4}+1\right)}{\left(\frac{s}{50}\right)^2+\frac{2(0.7)}{50}s+1} \right\}$$

Controller Transfer Function

The Bode diagram for $G(j\omega)$ is shown in Fig. 1. The closed-loop system has three regions of interest defined by

$$|G(j\omega)| \gg 1, \text{ over which } \left|\frac{G(j\omega)}{1 + G(j\omega)}\right| \doteq 1$$

$$|G(j\omega)| \ll 1, \text{ over which } \left|\frac{G(j\omega)}{1 + G(j\omega)}\right| \doteq |G(j\omega)| \quad \text{and}$$

$$|G(j\omega)| \doteq 1$$

The form of the closed loop transfer function, $\left|\frac{G(j\omega)}{1 + G(j\omega)}\right|$, in this last region defines the "dominant modes" of the closed-loop system dynamic response for impulse and step inputs. In most cases $G(j\omega)/[1 + G(j\omega)]$ in the region where $|G(j\omega)|$ is of the order of unity can be approximated by a first-, second-, or third-order system, the modes of which will determine the major features of the response. The open-loop amplitude asymptotes of an appropriate equivalent system for this example are shown in Fig. ..

Applying this approximation to the present example yields the closed-loop $(j\omega)$ Bode diagram of Fig. 2. The Bode diagram for the exact closed-loop system is also shown for comparative purposes. It will be observed that the error of the approximation is small. If greater accuracy is required, more complicated open-loop equivalent systems can be produced by retaining more of the terms in the complete open-loop transfer function.

In the example cited, the servo break frequency is of the order of ω_c. More typically, this frequency will be $\gg \omega_c$; the effect of the associated high-frequency leads and lags can then be approximated by replacing them in either the open- or closed-loop transfer functions by a pure time delay term, $e^{-\tau s}$. A satisfactory approximation for the time delay is $\tau = -(T_{leads} - T_{lags})$high frequency. (Alternative approximations are discussed in Chapter IV.)

In general, airplane transfer function break frequencies and time constants are spaced so that $G(j\omega)$ in the region of crossover can be satisfactorily approximated by a system of not more than fourth order.

The equivalent system concept can be applied to form numerical measures of each of the aspects of dynamic performance listed at the beginning of this chapter. However, determination of stability is usually only slightly more

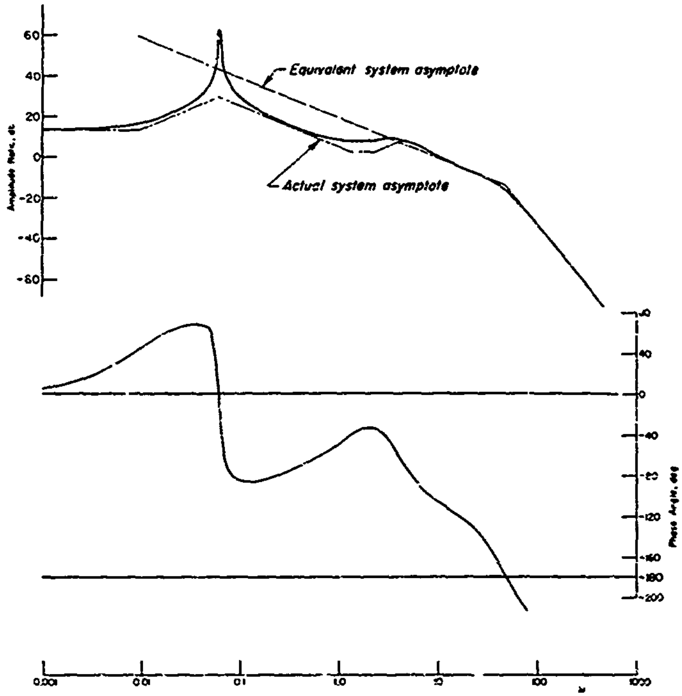

Figure 1. Open-Loop Bode Diagram of G(jω) Transfer Function

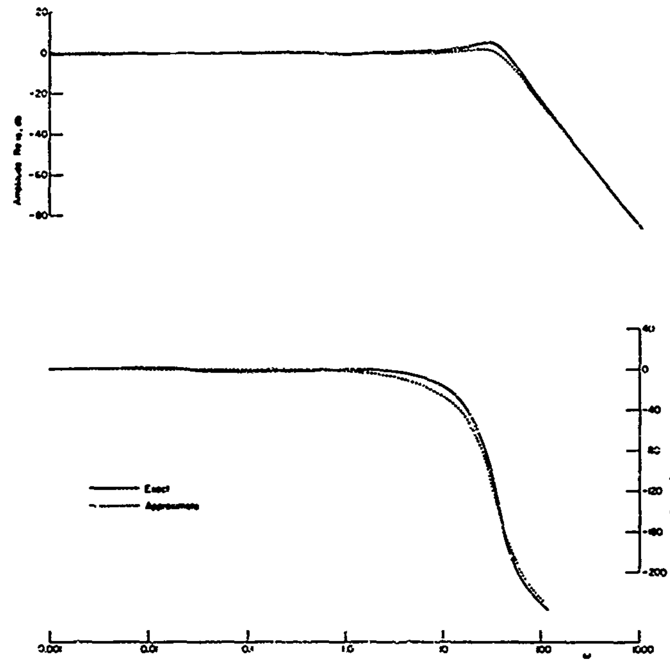

Figure 2. Comparison of Exact and Approximate Closed-Loop Bode Diagrams of $\frac{G(j\omega)}{1 + G(j\omega)}$ Transfer Function.

complicated for the complete linearized system model than for the simplified equivalent. Table I summarizes techniques for investigating stability. Since the meaningful application of almost all standard performance criteria requires that the system shall be stable, the use of one of the tests listed in Table I forms a prerequisite to the application of more refined criteria.

Of the remaining aspects of dynamic performance listed at the beginning of this chapter, this report is principally concerned with Item 2, "response to desired inputs," and the main body of the work presented here is concerned with the interpretation of this quality in numerical terms as a performance measure. Investigation of the system's capacity to suppress most unwanted inputs requires consideration of gusts, noise, etc., which are best described in statistical terms, and thus fall outside the scope of the present report. However, some unwanted inputs (e.g., engine failure) are well-described by deterministic expressions; for these inputs the methods of the present report are quite applicable. The topics of accuracy, insensitivity to parameter changes, and power/energy demands are discussed in Chapter VI in terms of desired performance measures.

Conventionally, the quantitative interpretation of the response to desired inputs is achieved by means of measures of the motion following a limited selection of deterministic inputs. These inputs are:

1. Impulse (Dirac delta function)
2. Step
3. Ramp
4. Power series
5. Sine wave.

Besides their conventional nature as test inputs, the first four of these are also representative of a wide variety of flight control system command, disturbance, or initial condition inputs.

The well-known convolution relationships of linear theory (e.g., Duhamel's integral), and specialized techniques described in Ref. 36 and 72, enable the response to any deterministic input to be calculated from a knowledge of the response to any of the inputs listed above. A comprehensive survey of performance measures that have been proposed for deterministic inputs is presented in the following two chapters.

TABLE I

STABILITY MEASURES FOR CONSTANT-COEFFICIENT

NAME	SYSTEM QUANTITIES INVOLVED	DIRECT MEASURES (ASSOCIATED MEASURES)
Descartes' Rule of Signs	Characteristic equation, $\Delta(s) = 0$	Number of variations in sign
Hakimi's Condition [40]	Characteristic equation, $\Delta(s) = 0$	Relative values of $\Delta(s)$ coefficients
Generalized Descartes' Rule of Signs	Modified characteristic equation, $\Delta(\lambda) = 0$; $\Delta(\lambda) = \Delta(s)\|_{s=\lambda-\sigma}$	Number of variations in sign
Routh-Hurwitz Criterion [22,64]	Characteristic equation, $\Delta(s) = 0$	Routh test functions; Hurwitz determinants
Generalized Routh-Hurwitz Criterion	Modified characteristic equation, $\Delta(\lambda) = 0$: $\Delta(\lambda) = \Delta(s)\|_{s=\lambda-\sigma}$	Routh test functions or Hurwitz determinants for modified characteristic equation in λ.
Liapounoff Theorem [19]	Matrix of coefficients, $[A]$; when system equations are in the form $[\dot{x}] = [A][x]$	Positive definite form of the symmetric matrix $[P]$
Mikhailov Criterion, Also [Leonhard or Cremer-Leonhard Criterion] [35,48,50,56]	Characteristic equation in form $\Delta(j\omega) = A(\omega^2) + j\omega B(\omega^2)$	Roots of $A(\omega^2)$ and $B(\omega^2)$
Cauchy-Nyquist Criterion [10,44,60]	Open-loop transfer function, $G(s) = \frac{\alpha(s)}{\beta(s)}$	Number of encirclements, N, of -1, and number of zeros, P, of $\beta(s) = 0$ in right half plane
		Gain margin, $K_{\text{neutral stability}}/K_{\text{actual}}$
		Phase margin, $\varphi(\omega_c) - \varphi_{\text{neutral stability}}$
		Peak magnification ratio, M_p
Generalized Nyquist Criterion [1,49,76]	Open-loop transfer function $G(s)$ $s = -\sigma \pm j\omega$ or $s = (-\zeta \pm j\sqrt{1-\zeta^2})\omega$	Number of encirclements of -1, and number of zeros in $\beta(s)$ with real parts greater than σ, or damping ratios less than ζ.
		Generalized gain margin, $\frac{K_{\|G(s)\|=1}}{K_{\text{actual}}}$
		Generalized phase margin
Precise Root Location		Actual root values of closed-loop system: ζ_i, ω_i, $1/T_i$.
1. Polynomial Factoring Techniques	Characteristic equation, $\Delta(s) = 0$	
2. Root Locus [27,28]	Poles, zeros, and gains of open-loop transfer function, $G(s)$	
3. Generalized $G(s)$ [49,53]	Open-loop transfer functions $G(s)$	

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TABLE I

STABILITY MEASURES FOR CONSTANT-COEFFICIENT LINEAR SYSTEMS

	DIRECT MEASURES (ASSOCIATED MEASURES)	APPLICATION TECHNIQUES AND AIDS; REMARKS
0	Number of variations in sign	All coefficients of same sign are a necessary, but not sufficient, condition for stability. Number of positive [negative] real roots cannot exceed the number of variations of sign of $\Delta(s)$, $[\Delta(-s)]$.
0	Relative values of $\Delta(s)$ coefficients	Necessary conditions for roots of $\Delta(s) = a_n s^n + a_{n-1} s^{n-1} + \dots + a_0$, $(a_i > 0)$, to be negative and real are: no local minima of the a_i's and min $[a_i]$ = min $[a_0, a_n]$.
.	Number of variations in sign	All coefficients of same sign are a necessary, but not sufficient, condition for all roots to have negative real parts less than assigned value of $-\sigma$.
0	Routh test functions; Hurwitz determinants	Provide necessary and sufficient conditions for stability; also determines number of roots in right half plane. Routh's Algorithm is a simple aid for development of higher-order test functions. One test function is critical when a parameter variation causes a change from stability to instability. (Ref. 22)
,	Routh test functions or Hurwitz determinants for modified characteristic equation in λ.	Provide necessary and sufficient conditions for roots to have real parts less than assigned values of $-\sigma$.
n	Positive definite form of the symmetric matrix $[P]$	System is asymptotically stable if the matrix $[P]$, which is symmetric ($[P]$ = Transpose $[P] = [P^T]$), satisfying the matrix equation $[A^T][P] + [P][A] = -[I]$ (the unit matrix) is positive definite (all principal minors of $[P]$ are positive). The elements of $[P]$ are found from the $n(n+1)/2$ simultaneous equations resulting from expanding the matrix equation above. The theorem can also be stated in terms of the Liapounoff function $V([x]) = [x^T][P][x]$.
	Roots of $A(\omega^2)$ and $B(\omega^2)$	Type and relative sequence of roots indicates condition of stability or instability. In stable situation roots of $A(\omega^2) = 0$ and $B(\omega^2) = 0$ are simple, real, and positive; and they alternate in the sequence: A B A B, etc.; usually applied graphically.
	Number of encirclements, N, of -1, and number of zeros, P, of $G(s) = 0$ in right half plane	Gives number of zeros, $Z = P + N$, of $1 + G(s) = 0$ in right half plane. Ordinarily applied graphically with polar plot or $G(j\omega)$ Bode diagrams.
	Gain margin, $K_{neutral\ stability}/K_{actual}$	Gain change necessary for neutral stability.
	Phase margin, $\varphi(\omega_c) - \varphi_{neutral\ stability}$	Phase change required for neutral stability, with gain held constant.
	Peak magnification ratio, M_p	Measures maximum closed-loop resonance. Usually determined from open-loop plots (polar or logarithmic gain-phase) and closed-loop overlays (M circles or Nichols chart).
	Number of encirclements of -1, and number of zeros in $G(s)$ with real parts greater than σ, or damping ratios less than ζ.	Gives number of zeros of $1 + G(s) = 0$ which have real parts greater than σ, or damping ratios less than ζ. Ordinarily applied with $G(s)$ Bode diagrams.
)u	Generalized gain margin, $\dfrac{K_{\lvert G(s)\rvert = 1}}{K_{actual}}$	Gain change required to achieve roots with specified σ or ζ.
	Generalized phase margin	Phase change required, with gain held constant, to achieve roots with specified σ or ζ.
0 loop	Actual root values of closed-loop system: ζ_i, ω_i, $1/T_i$.	Complete definition of system transfer characteristics. With knowledge of input, response is completely defined. Techniques include: Newton's method, Horner's method, synthetic division, Graeffe's root-squaring method, Lin's method, etc. Supplementary Techniques in Unified Servo Analysis. (Ref. 33)

2

CHAPTER II

SYSTEM CHARACTERISTICS

The term "system characteristics" will be used to denote all performance measures not expressed as explicit functions of error. For example, phase margin and time-to-peak are both system characteristics, whereas ITAE is classified as an "indicial error measure" and is discussed in Chapter III. Although an infinite number of system characteristics could be devised, useful performance measures in practice are obtained only from characteristics directly descriptive of the system transfer function or the impulsive or indicial (step-input) response.

Tables II-A and II-B summarize a broad cross-section of system characteristics. All the characteristics that have been found in the references listed at the end of this report are included, with one exception (the product of peak overshoot and time-to-peak), which is discussed on page 27. The characteristics listed in Table II-A are quantities directly obtainable from $G(s)$, $G(j\omega)$, or the closed loop forms $\frac{G(s)}{1 + G(s)}$, $\frac{G(j\omega)}{1 + G(j\omega)}$. It is possible to generalize some of the $G(j\omega)$ measures, such as phase margin and gain margin, into analogous quantities for $G(s)$ forms. Apart from the error coefficients, the remaining entries on Table II-B describe the response to a step input, and will be called "indicial response characteristics."

Ideally, the indicial response characteristics would partition the response into the regions indicated in Fig. 3. "Dead time" is the time to attain 10 percent of the final value, rise time is the time to go from 10 percent to 90 percent, and decay time is the time for the transient to decay from 90 percent to within 5 percent of the final value. The sum of the dead time, rise time, and decay time is known as the settling time (which can be defined directly as the time for the indicial response to reach and thereafter remain within 5 percent of the final value). Unfortunately, only the settling and rise times are readily related to transfer function characteristics, so the quantities involved in the idealized partitioning of the indicial response are replaced by the somewhat overlapping measures indicated on Table II-B.

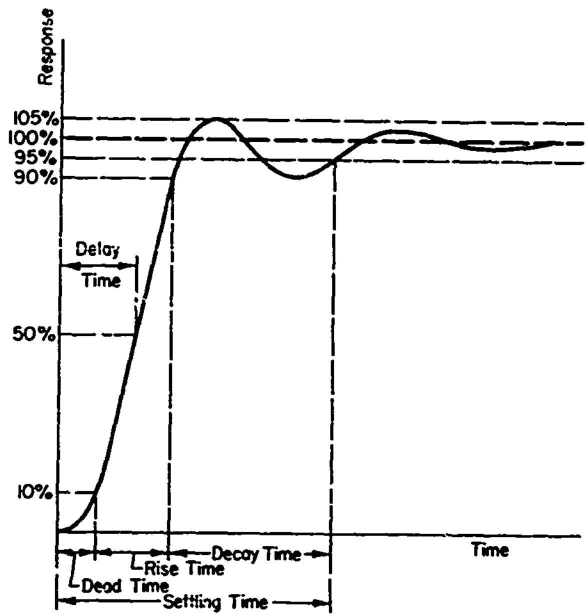

Figure 3. Idealized Partitioning of Indicial Reponse

TABLE II-A

SINGLE-LOOP FEEDBACK SYSTEM CHARACTERISTICS

TRANSFER FUNCTION MEASURES

SYSTEM CHARACTERISTIC	DEFINITION	APPLICATION TECHNIQUES AND AIDS; REMARKS	FORMULA FOR UNIT NUMERATOR SECOND-O[RDER] IN TERMS OF OPEN LOOP PARAMETERS	IN TERMS OF [...]
Open-loop transfer function, $G(s)$	1. Ratio of Laplace transforms of output and error, $G(s) = \frac{C(s)}{E(s)}$ 2. Complex coefficient of e^{st} component of output response, $c(t)$, when error $= e^{st}$	Specialized values can be associated with either special values of s or with the forced component (particularly integral) of the output response to error inputs. **Special Value of s** — **Special Value of G(s)** $s = j\omega$ — $G(j\omega) = \lvert G(j\omega)\rvert e^{j\angle G(j\omega)}$ $s = \sigma$ — $G(\sigma)$ $s = \left(-\zeta \pm j\sqrt{1-\zeta^2}\right)\omega$ — $G(\zeta,\omega) = \lvert G(\zeta,\omega)\rvert e^{j\angle G(\zeta,\omega)}$ — **Forced Component of c(t)** $e^{j\omega t}$ — $\lvert G(j\omega)\rvert e^{j\left[\omega t + \angle G(j\omega)\right]}$ $\cos \omega t$ — $\lvert G(j\omega)\rvert \cos\left[\omega t + \angle G(j\omega)\right]$ $e^{\sigma t}$ — $G(\sigma)e^{\sigma t}$ $e^{-\left(\zeta \pm j\sqrt{1-\zeta^2}\right)\omega t}$ — $\lvert G(\zeta,\omega)\rvert e^{-\zeta\omega t \pm j\left[\sqrt{1-\zeta^2}\,\omega t + \angle G(\zeta,\omega)\right]}$	$\frac{K}{s(s+\lambda)}$	
Closed-loop transfer functions, $G_{Ec}(s)$, $G_{Cc}(s)$	$G_{Ec}(s) = \frac{E(s)}{R(s)} = \frac{1}{1+G(s)}$ $G_{Cc}(s) = \frac{C(s)}{R(s)} = \frac{G(s)}{1+G(s)}$	Precise location of closed-loop system poles and zeros determined from open-loop transfer function representation by URAM, or its constituent techniques.	$\frac{K}{s^2 + \lambda s + K}$	[illegible]
Open-loop frequency domain measures				
Crossover frequency, ω_c	Frequency where $\lvert G(j\omega)\rvert = 1$	Determined from open-loop $G(j\omega)$ Bode diagram	$\frac{1}{\sqrt{2}}\left[\sqrt{\lambda^4 + 4K^2} - \lambda^2\right]^{1/2}$	$\omega_n\left(\ldots\right.$
Phase margin, ϕ_m	$\phi_1 - \angle G(j\omega_c)$ where ϕ_1 is the phase angle for instability (usually 180 deg)		$\tan^{-1}\left[\frac{2}{\sqrt{\frac{4K^2}{\lambda^4} + 1} - 1}\right]^{1/2}$	$\tan^{-1}\ldots$
Gain margin, K_1/K	Ratio of open-loop gain required to give instability to actual gain		∞	
Frequency of instability, ω_1	Frequency at which instability occurs		∞	
Open-loop ζ,ω domain measures				
Generalized crossover frequency, ω_c	Value of ω for $\lvert G(\zeta,\omega)\rvert = 1$	Determined from open-loop $G(\zeta,\omega)$ Bode diagram		
Generalized phase margin, ϕ_{m_ζ}	$\phi_1 - \angle G(\zeta,\omega_c)$, where ϕ_1 is the phase angle required for a closed-loop roots with damping ratio ζ (usually 180 deg)			
Generalized gain margin, K_1/K	Ratio of open-loop gain required to give closed-loop roots with damping ratio ζ to actual gain			
Closed-loop frequency domain measures				
Deviation ratio	$\lvert G_{Ec}(j\omega)\rvert = \left\lvert\frac{E(j\omega)}{R(j\omega)}\right\rvert$	Determined from difference of open- and closed-loop Bodes $\left\lvert\frac{E}{R}\right\rvert_{db} = \lvert G_{Cc}\rvert_{db} - \lvert G\rvert_{db}$, or by direct calculation	$\left[\frac{\omega^4 + \lambda^2\omega^2}{(K-\omega^2)^2 + \lambda^2\omega^2}\right]^{1/2}$	$\left[\frac{\ldots}{(\omega_n^2 \ldots}\right.$
Peak magnification frequency, ω_p	Frequency at which maximum $\lvert C/R(j\omega)\rvert$ occurs		$\sqrt{K - \frac{\lambda^2}{2}}$	
Peak magnification ratio, M_p	Maximum value of $\lvert C/R(j\omega)\rvert$		$\frac{2K}{\lambda\sqrt{4K - \lambda^2}}$	
Bandwidth, ω_b	Highest frequency for which $\lvert C/R(j\omega)\rvert_{db}$ is less than 3 db below $\lvert C/R(0)\rvert_{db}$		$\left[K - \frac{\lambda^2}{2} + \sqrt{2K^2 - \lambda^2 K + \frac{\lambda^4}{4}}\right]^{1/2}$	$\omega_n\left(1 - 2\zeta^2 \ldots\right.$

TABLE II-A

SINGLE-LOOP FEEDBACK SYSTEM CHARACTERISTICS

TRANSFER FUNCTION MEASURES

<table>
<tr><th rowspan="2">APPLICATION TECHNIQUES AND AIDS; REMARKS</th><th colspan="2">FORMULA FOR UNIT NUMERATOR SECOND-ORDER SYSTEM</th><th rowspan="2">REF</th></tr>
<tr><th>IN TERMS OF OPEN-LOOP PARAMETERS</th><th>IN TERMS OF CLOSED-LOOP PARAMETERS</th></tr>
<tr><td>Specialized values can be associated with either special values of s or with the forced component (particularly integral) of the output response to error inputs.
<table>
<tr><th>Special Value of s</th><th>Special Value of G(s)</th></tr>
<tr><td>$s = j\omega$</td><td>$G(j\omega) = |G(j\omega)| e^{j\angle G(j\omega)}$</td></tr>
<tr><td>$s = \sigma$</td><td>$G(\sigma)$</td></tr>
<tr><td>$s = \left(-\zeta \pm j\sqrt{1-\zeta^2}\right)\omega$</td><td>$G(\zeta,\omega) = |G(\zeta,\omega)| e^{j\angle G(\zeta,\omega)}$</td></tr>
</table>
<table>
<tr><th></th><th>Forced Component of c(t)</th></tr>
<tr><td>$e^{j\omega t}$</td><td>$|G(j\omega)| e^{j\left[\omega t + \angle G(j\omega)\right]}$</td></tr>
<tr><td>$\cos \omega t$</td><td>$|G(j\omega)| \cos\left[\omega t + \angle G(j\omega)\right]$</td></tr>
<tr><td>$e^{\sigma t}$</td><td>$G(\sigma)e^{\sigma t}$</td></tr>
<tr><td>$e^{-\left(\zeta \pm j\sqrt{1-\zeta^2}\right)\omega t}$</td><td>$|G(\zeta,\omega)| e^{-\zeta\omega t \pm j\left[\sqrt{1-\zeta^2}\,\omega t + \angle G(\zeta,\omega)\right]}$</td></tr>
</table></td><td>$\frac{K}{s(s+\lambda)}$</td><td>$\frac{\omega_n^2}{s(s+2\zeta\omega_n)}$</td><td></td></tr>
<tr><td>Root or location of closed-loop system poles and zeros determined from open-loop transfer function representation by URAM, or its constituent techniques.</td><td>$\frac{K}{s^2+\lambda s+K}$</td><td>$\frac{\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2}$</td><td></td></tr>
<tr><td>Determined from open-loop G(jω) Bode diagram</td><td>$\frac{1}{\sqrt{2}}\left[\sqrt{\lambda^4+4K^2}-\lambda^2\right]^{1/2}$
$\tan^{-1}\left[\frac{2}{\sqrt{\frac{4K^2}{\lambda^4}+1}-1}\right]^{1/2}$
.
.</td><td>$\omega_n\left(\sqrt{4\zeta^4+1}-2\zeta^2\right)^{1/2}$
$\tan^{-1}\frac{2\zeta}{\left(\sqrt{4\zeta^4+1}-2\zeta^2\right)^{1/2}}$
.
.</td><td>63</td></tr>
<tr><td>Determined from open-loop G(ζ,ω) Bode diagram</td><td></td><td></td><td></td></tr>
<tr><td>Determined from difference of open- and closed-loop Bodes $\left|\frac{E}{R}\right|_{db} = |G_{OL}|_{db} - |G_{CL}|_{db}$, or by direct calculation</td><td>$\left[\frac{\omega^4+\lambda^2\omega^2}{(K-\omega^2)^2+\lambda^2\omega^2}\right]^{1/2}$
$\sqrt{K-\frac{\lambda^2}{2}}$
$\frac{2K}{\lambda\sqrt{4K-\lambda^2}}$
$\left[K-\frac{\lambda^2}{2}+\sqrt{2K^2-\lambda^2K+\frac{\lambda^4}{4}}\right]^{1/2}$</td><td>$\left[\frac{\omega^4+4\zeta^2\omega_n^2\omega^2}{(\omega_n^2-\omega^2)^2+4\zeta^2\omega_n^2\omega^2}\right]^{1/2}$
$\omega_n\sqrt{1-2\zeta^2}$
$\frac{1}{2\zeta\sqrt{1-\zeta^2}}$
$\omega_n\left(1-2\zeta^2+\sqrt{2-4\zeta^2+4\zeta^4}\right)^{1/2}$</td><td>63
1
63
72</td></tr>
</table>

2

TABLE II-3

TRANSIENT RESPONSE CHARACTERISTICS

INPUT	NAME	DEFINITION	FORMULAE FOR SECOND-ORDER UNIT-NUMERATOR SYSTEM: IN TERMS OF OPEN-LOOP PARAMETERS κ, and λ	IN TERMS OF CLOSE[D-LOOP] PARAMETERS ζ, a[nd …]
Step	Final Value	$\lim_{t\to\infty} c(t) = \lim_{s\to 0} s\,C(s)$	1	1
	Settling Time, t_s	Time for indicial response to first attain and remain within 5 percent of final value	$= -\frac{1}{\lambda}\ln\left(0.0025 - \frac{\lambda^2}{1600\kappa}\right)$	$\doteq \frac{\ln\left(0.05\sqrt{1-\zeta^2}\right)}{-\zeta\omega_n}$ $\doteq \frac{3}{\zeta\omega_n}$, for $\zeta < 0.9$
	Equivalent Time Constant, T_{eq}	Time for indicial response to first achieve 63 percent of final value	$\doteq \frac{1}{\omega_c} - \frac{\sqrt{2}}{\left(\sqrt{\lambda^4 + 4\kappa^2} - \lambda^2\right)}$	$\doteq \frac{1}{\omega_c} - \frac{1}{\omega_n\left(\sqrt{4\zeta^4 + 1} - 2 \ldots\right.}$
	Delay Time, t_d	Time for indicial response to first reach 50 percent of final value	$\doteq \frac{1}{\sqrt{\kappa}} + 0.35\frac{\lambda}{\kappa}$	$\doteq \frac{1 + 0.7\zeta}{\omega_n}$
	Time to Peak, t_p	Time for indicial response to achieve its maximum value	$\frac{2\pi}{\sqrt{4\kappa - \lambda^2}}$	$\frac{\pi}{\omega_n\sqrt{1-\zeta^2}}$
	Rise Time, t_r	Time for indicial response to rise from 10 to 90 percent of its final value	$\doteq 1.76\frac{\lambda}{\kappa} + \frac{0.2}{\lambda}$	$\doteq \frac{7.04\zeta^2 - 0.2}{2\zeta\omega_n}$
	Peak Overshoot, c_p	Maximum value of indicial response	$1 + e^{-\frac{\lambda\pi}{\sqrt{4\kappa-\lambda^2}}}$	$1 + e^{-\frac{\zeta\pi}{\sqrt{1-\zeta^2}}}$
Power Series	Dynamic Error Coefficients $E_0, E_1, E_2, \ldots$	$\frac{E(s)}{R(s)} = E_0 + E_1 s + E_2 s^2 + \ldots$	$E_n = \frac{1}{n!}\lim_{s\to 0}\frac{d^n}{ds^n}\left[\frac{1}{1 + \frac{\kappa}{s(s+\lambda)}}\right]$	$E_1 = \frac{1}{n!}\lim_{s\to 0}\frac{d^n}{ds^n}\left[\frac{\ldots}{1 + \frac{\ldots}{s(\ldots}}\right.$

1

TABLE II-B

TRANSIENT RESPONSE CHARACTERISTICS

[illegible]EFINITION	FORMULAE FOR SECOND-ORDER UNIT-NUMERATOR SYSTEM		REMARKS
	IN TERMS OF OPEN-LOOP PARAMETERS κ, and λ	IN TERMS OF CLOSED-LOOP PARAMETERS ζ, and ω_n	
$\lim_{s\to 0} sC(s)$	1	1	
[illegible]al response to [illegible] reach within [illegible]al value	$\approx \frac{1}{\lambda}\ln\left(0.0025 - \frac{\lambda^2}{1000\kappa}\right)$	$\approx \frac{\ln\left(0.05\sqrt{1-\zeta^2}\right)}{-\zeta\omega_n}$ $\approx \frac{3}{\zeta\omega_n}$, for $\zeta < 0.5$	See Fig. 5. For first-order dominant mode $t_s = 3T_{eq}$ See Fig. 6 for comparison of exact values with this approximation applied to a second-order system. (Ref. 36)
[illegible]al response to [illegible]5 percent of	$\approx \frac{1}{\omega_c} = \frac{\sqrt{2}}{\left(\sqrt{\lambda^4 + 4\kappa^2} - \lambda^2\right)}$	$\approx \frac{1}{\omega_c} = \frac{1}{\omega_n\left(\sqrt{4\zeta^4+1} - 2\zeta^2\right)^{1/2}}$	Approximation $T_{eq} = \frac{1}{\omega_c}$ is valid when first-order dominant mode exists. Approximation is unsuitable for second-order systems. (See Fig. 7)
[illegible]al response to [illegible]percent of final	$\approx \frac{1}{\sqrt{\kappa}} + 0.35\frac{\lambda}{\kappa}$	$\approx \frac{1 + 0.7\zeta}{\omega_n}$	See Fig. 7
[illegible]al response to [illegible]imum value	$\frac{2\pi}{\sqrt{4\kappa - \lambda^2}}$	$\frac{\pi}{\omega_n\sqrt{1-\zeta^2}}$	
[illegible]al response to [illegible] 90 percent o[illegible]	$\approx 1.76\frac{\lambda}{\kappa} + \frac{0.2}{\lambda}$	$\approx \frac{7.04\zeta^2 + 0.2}{2\zeta\omega_n}$	Accurate within ±20 percent for $0.1 < \zeta < 1.0$
[illegible]f indicial	$1 + e^{-\frac{\lambda\pi}{\sqrt{4\kappa-\lambda^2}}}$	$1 + e^{-\frac{\zeta\pi}{\sqrt{1-\zeta^2}}}$	
[illegible] $+ E_2 s^2 + \ldots$	$E_n = \frac{1}{n!}\lim_{s\to 0}\frac{d^n}{ds^n}\left[\frac{1}{1 + \frac{\kappa}{s(s+\lambda)}}\right]$	$E_n = \frac{1}{n!}\lim_{s\to 0}\frac{d^n}{ds^n}\left[\frac{1}{1 + \frac{\omega_n^2}{s(s+2\zeta\omega_n)}}\right]$	The error coefficients can be interpreted in terms of time-weighted integrals of the impulsive response. (Ref. 72 and Chapter V) $E_n = (-1)^n \frac{1}{n!}\int_0^\infty t^n e(t)dt.$ For assessment as criterion see text and Table 5. (Ref. 11)

2

In order to assess the merit of any given system dynamic performance from its indicial response, it is necessary to define "good" dynamic performance in terms of the indicial response. It is generally agreed that "good" dynamic performance implies low overshoot, short dead time, fast rise time, and good damping of motions subsequent to the decay time. (The last requirement implies low settling time.) Thus a good indicial response will comply with certain specifications on its "shape," which can be defined in terms of overshoot, ratios of dead-time to settling-time and rise-time, etc. However specifications of "shape" alone is insufficient to ensure that the indicial response will be satisfactory; some parameter defining the time scale of the response must also be specified. The implications of this last requirement will now be discussed briefly.

The over-all time scale of the indicial response depends upon the bandwidth of the system. Practical considerations of inertia, power demands, etc., result in increasing penalties in weight and complication as bandwidth is increased. However, in this generalized investigation it is not possible to set these upper limits upon bandwidth explicitly. For statistically described inputs and for deterministic inputs (such as rectangular pulses) which are of finite specified durations, lower limits on bandwidth can be set at least approximately. For example, the settling time should not exceed the pulse duration. However, for impulses and step inputs, upper limits on settling time cannot be specified in the absence of further information regarding the operating environment. Thus this chapter and Chapter III are essentially limited to a study of those aspects of dynamic performance which can be represented by the "shape" of the indicial response. To focus attention on "shape" rather than time scale, performance measures such as settling-time, rise-time, etc., are all expressed in nondimensional forms. For example, second-order system characteristics such as rise-time, settling-time, etc., are normalized through multiplication by ω_n, where ω_n is the system undamped natural frequency. A more general procedure for normalization will be given at the beginning of Chapter III.

The general procedure for obtaining system characteristics is to construct the appropriate transfer function representation or indicial response, and read the measure directly. Thus, crossover frequency, phase and gain margins, and frequency of instability may be obtained from the open-loop $G(j\omega)$ Bode diagram. Application of the unified servo analysis method of Ref. 53 (hereafter referred

to as U.S.A.M.) facilitates swift construction of the closed-loop $G(j\omega)$ Bode diagram, inspection of which yields peak magnification ratio and frequency, and bandwidth.

For second-order systems, it is possible to develop exact formulae for many system characteristics and approximations for the remainder. Tables II-A and II-B list these formulae in terms of open- and closed-loop parameters for a system having the open-loop transfer function.

$$G(s) = \frac{\kappa}{s(s + \lambda)} \tag{2}$$

and a closed-loop transfer function,

$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} \tag{3}$$

where ω_n is the undamped natural frequency, and ζ is the damping ratio. The closed- and open-loop parameters are related by the equations

$$\kappa = \omega_n^2 \tag{4}$$

$$\lambda = 2\zeta\omega_n \tag{5}$$

(This second-order system corresponds to a wide variety of equivalent systems encountered in flight control applications). Elementary operations on these relations enable exact formulae to be found for all transfer function frequency domain characteristics. The resulting formulae are listed in Table II-A, together with references where derivations can be found. Simpler approximate formulae may sometimes be preferred, and two standard approximations are discussed below.

Figure 4 illustrates a linear approximation to phase margin

$$\zeta = \frac{1}{2} \times \frac{\varphi_m}{57.3} \tag{6}$$

Comparison with the exact result indicates acceptable accuracy for $\varphi_m < 50$ deg. Bandwidth may also be found approximately using

$$\omega_b = (1 + \sqrt{2})\omega_c \tag{7}$$

which is obtained by neglecting terms in ζ in the exact expression in Table III. Equation 7 is compared with the exact bandwidth for a second-order system in Fig. 5.

Phase margin, bandwidth, and peak magnification ratio are widely used for performance specification. The phase margins of all the standard forms presented in Chapter III (Table VII) are between 50 and 70 deg. In systems with a dominant second-order mode, this would be expected to yield adequate ($\zeta \doteq 0.7$) damping. Similarly, elimination of the frequency response peak leads to adequate damping of the dominant modes. The connection of these measures with the transient response is, in general, neither unique nor explicit, except for second order [illegible] as subsequently discussed in connection with Table III. A good system for a given application must of necessity have phase margin, bandwidth, and peak magnification values which lie within relatively narrow limits, but a system which complies with these limits is not necessarily good. This follows, of course, from the fact that the behavior of the actual transfer functions of interest is defined only in a gross sense over a narrow frequency band by these particular measures. In general, therefore, none of these characteristics taken alone yields a valid, selective, and reliable measure.

Of the indicial response characteristics listed in Table II-B, exact formulae exist only for time-to-peak and peak overshoot of second-order systems. For higher-order systems, time-to-peak, settling time, and rise time can be estimated for some special classes of systems by means of the charts of Ref. 13, 17, and 24. Reference 13 presents rise times for several classes of third-order systems; the results are discussed on page 31. The rest of this chapter is mainly devoted to a discussion of the calculation and interrelation of second-order system characteristics.

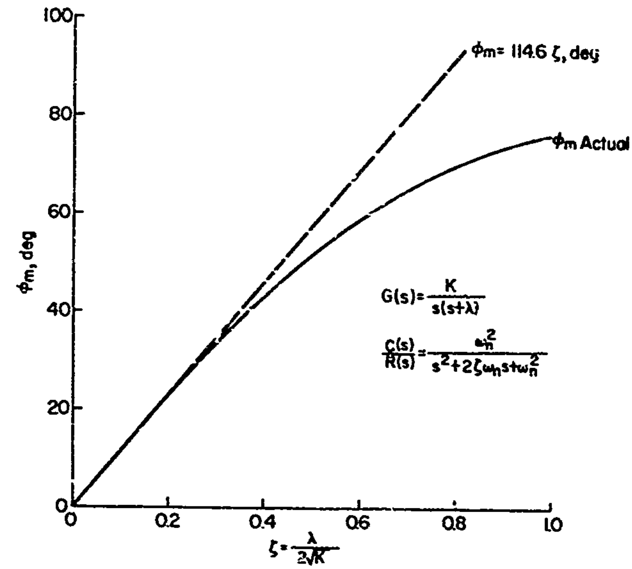

Figure 4. Comparison of Actual and Approximate Phase Margins

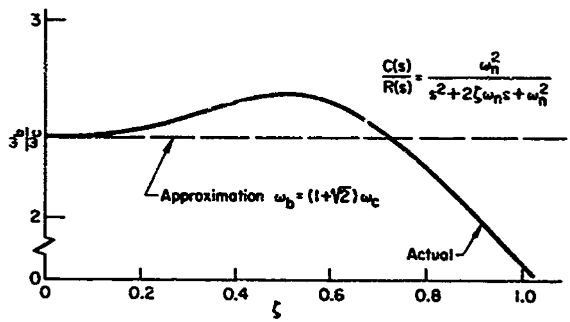

Figure 5. Comparison of Actual and Approximate Bandwidths

1

	ω_c	φ_m
ω_c	1	$2\zeta\omega_n \cot \varphi_m$
φ_m	$\tan^{-1} \frac{2\zeta\omega_n}{\omega_c}$	1
ω_p^2	$\omega_c^2 \frac{(1 - 2\zeta^2)}{\left(\sqrt{4\zeta^4 + 1} - 2\zeta^2\right)}$	$\omega_n^2 \left(1 - \frac{\sin \varphi_m \tan \varphi_m}{2}\right)$
M_p	$\frac{2\omega_n^2\omega_c^2}{\sqrt{4\omega_n^2\omega_c^2(\omega_n^4 - \omega_c^4) - (\omega_n^4 - \omega_c^4)^2}}$	$\frac{2}{\tan \varphi_m \sqrt{4 \cos \varphi_m - \sin^2 \varphi_m}}$
ω_b^2	$\omega_c^2 \frac{\left(1 - 2\zeta^2 + \sqrt{2 - 4\zeta^2 + 4\zeta^4}\right)}{\left(\sqrt{4\zeta^4 + 1} - 2\zeta^2\right)}$	$\omega_n^2 \left\{1 - \frac{\sin \varphi_m \tan \varphi_m}{2} + \sqrt{2 - \sin \varphi\right.$
t_d (Approx.)	$\doteq \frac{1}{\omega_c}\left(1 + 0.7\zeta\right)\left(\sqrt{4\zeta^4 + 1} - 2\zeta^2\right)^{1/2}$	$\doteq \frac{1 + 0.35 \tan \varphi_m \cos^{1/2} \varphi_m}{\omega_n}$
t_p	$\frac{2\pi\omega_c}{(\omega_c^4 + 4\omega_n^2\omega_c^2 - \omega_n^4)^{1/2}}$	$\frac{2\pi}{\omega_n(4 - \sin \varphi_m \tan \varphi_m)^{1/2}}$
c_p	$1 + e^{-\pi\left(\frac{\omega_n^4 - \omega_c^4}{4\omega_n^2\omega_c^2 - \omega_n^4 + \omega_c^4}\right)^{1/2}}$	$1 + e^{\frac{-\pi \tan \varphi_m}{\sqrt{4 \sec \varphi_m - \tan^2 \varphi_m}}}$

φ_m	ω_p	
$\cot\varphi_m$	$\omega_n^2 - \omega_p^2 + \sqrt{2\omega_n^4 - 2\omega_n^2\omega_p^2 + \omega_p^4}$	$\frac{\omega_n^2}{M_p}$
	$\tan^{-1}\left[\frac{2(\omega_n^2 - \omega_p^2)}{(2\omega_n^4 - 2\omega_n^2\omega_p^2 + \omega_p^4)^{1/2} + \omega_p^2 - \omega_n^2}\right]^{1/2}$	ta
$- \frac{\sin\varphi_m \tan\varphi_m}{2}\Big)$	1	$\frac{\omega_p^2}{M_p}$
$\frac{2}{\varphi_m\sqrt{4\cos\varphi_m - \sin^2\varphi_m}}$	$\frac{\omega_n^2}{(\omega_n^4 - \omega_p^4)^{1/2}}$	1
$1 - \frac{\sin\varphi_m\tan\varphi_m}{2} + \sqrt{2 - \sin\varphi_m\tan\varphi_m + \frac{\sin^2\varphi_m\tan^2\varphi_m}{4}}\Big)$	$\omega_p^2 + (\omega_n^4 + \omega_p^4)^{1/2}$	$\frac{\omega_n^2}{M_p}$
$\frac{+\,0.35\tan\varphi_m\cos^{1/2}\varphi_m}{\omega_n}$	$\doteq \frac{1}{\omega_n}\left(1 + 0.495\sqrt{1 - \frac{\omega_p^2}{\omega_n^2}}\right)$	$\doteq$
$\frac{2\pi}{(- \sin\varphi_m\tan\varphi_m)^{1/2}}$	$\frac{\pi\sqrt{2}}{\sqrt{\omega_n^2 + \omega_p^2}}$	ω_n
$e^{\frac{-\pi\tan\varphi_m}{\sqrt{4\sec\varphi_m - \tan^2\varphi_m}}}$	$1 + e^{-\pi\left[\frac{\omega_n^2 - \omega_p^2}{\omega_n^2 + \omega_p^2}\right]^{1/2}}$	1

TA

CORRELATION OF SECOND-ORDER UNIT-N

p	M_p	
$\omega_p^2 + \omega_p^4$	$\frac{\omega_n^2}{M_p}\left\{\left[3M_p^2 - 2M_p\sqrt{M_p^2 - 1} - 1\right]^{1/2} - M_p + \sqrt{M_p^2 - 1}\right\}$	$\omega_n^2 \left\{\left[\frac{\omega_n^2}{4\omega_p^4}\right.\right.$
$\left. \frac{- \omega_p^2)}{\omega_p^4)^{1/2} + \omega_p^2 - \omega_n^2}\right]^{1/2}$	$\tan^{-1} \frac{\sqrt{2}}{M_p}\left\{\left(M_p - \sqrt{M_p^2 - 1}\right)^2 + \left(M_p - \sqrt{M_p^2 - 1}\right)\left(3M_p^2 - 2M_p\sqrt{M_p^2 - 1} - 1\right)^{1/2}\right\}^{1/2}$	$\tan^{-1}\left\{\frac{1}{2}\right.$
	$\frac{\omega_n^2}{M_p}\sqrt{M_p^2 - 1}$	$\frac{1}{2\omega_p^2}\left(\omega_p^4\right.$
	1	$\frac{3}{\left(6 - \frac{\omega_n^4}{\omega_p^4}\right.}$
	$\frac{\omega_n^2}{M_p}\left[\sqrt{M_p^2 - 1} + \sqrt{2M_p^2 - 1}\right]$	1
$\left. - \frac{\omega_p^2}{\omega_n^2}\right)$	$\doteq \frac{1}{\omega_n}\left[1 + 0.7\left(\frac{M_p - \sqrt{M_p^2 - 1}}{2M_p}\right)^{1/2}\right]$	$\doteq \frac{(1 +}{\omega}$
	$\frac{\pi\sqrt{2M_p}}{\omega_n\left(M_p + \sqrt{M_p^2 - 1}\right)^{1/2}}$	$\frac{2\pi}{\omega_n}\frac{}{\left(\frac{\omega_p^2}{\omega_n^2} + \right.}$
	$1 + e^{-\pi\left[\frac{M_p - \sqrt{M_p^2 - 1}}{M_p + \sqrt{M_p^2 - 1}}\right]^{1/2}}$	$1 + e^x ,$

3

TABLE III

…LATION OF SECOND-ORDER UNIT-NUMERATOR SYSTEM CHARACTERISTICS

	M_p	ω_b
	$\ldots M_p^2 - 1\Big\}$	$\omega_n^2\left\{\left[\frac{\omega_n^2}{4\omega_b^4} + \frac{\omega_n^2}{\omega_b^2} + \frac{1}{2} - \frac{\omega_b^2}{\omega_n^2} + \frac{\omega_b^4}{4\omega_n^4}\right]^{1/2} - \frac{\omega_n^2}{2\omega_b^2} - 1 + \frac{\omega_b^2}{2\omega_n^2}\right\}$
	$\ldots - 1\Big)\left(3M_p^2 - 2M_p\sqrt{M_p^2 - 1} - 1\right)^{1/2}\Big\}^{1/2}$	$\tan^{-1}\left\{\frac{1}{2}\left[\frac{\omega_n^2}{\omega_b^2} + 2 - \frac{\omega_b^2}{\omega_n^2}\right]^2 + \left(\frac{\omega_n^2}{\omega_b^2} + 2 - \frac{\omega_b^2}{\omega_n^2}\right)\left[\frac{1}{4}\left(\frac{\omega_n^2}{\omega_b^2} + 2 - \frac{\omega_b^2}{\omega_n^2}\right)^2 + 1\right]^{1/2}\right\}^{1/2}$
		$\frac{1}{2\omega_b^2}\left(\omega_b^4 - \omega_n^4\right)$
		$\frac{2}{\left(6 - \frac{\omega_n^4}{\omega_b^4} - \frac{\omega_b^4}{\omega_n^4}\right)^{1/2}}$
		1
		$\doteq \frac{(1 + 0.7\zeta)}{\omega_b}\left(1 - 2\zeta^2 + \sqrt{2 - 4\zeta^2 + 4\zeta^4}\right)^{1/2}$
		$\frac{2\pi}{\omega_n}\frac{1}{\left(\frac{\omega_n^2}{\omega_b^2} + 2 - \frac{\omega_b^2}{\omega_n^2}\right)^{1/2}}$
		$1 + e^{\chi}$, where $\chi = -\tau\left[\frac{\frac{\omega_n^2}{\omega_b^2} + 2 - \frac{\omega_b^2}{\omega_n^2}}{\frac{\omega_r^2}{\omega_b^2} + 2 - \frac{\omega_b^2}{\omega_r^2}}\right]^{1/2}$

4

t_d (Approx.)	t_p
$\doteq \left(\frac{1+0.7\zeta}{t_d}\right)^2\left(\sqrt{4\zeta^4+1}-2\zeta^2\right)$	$\frac{\pi^2}{t_p^2(1-\zeta^2)}\left[\sqrt{4\zeta^4+1}-2\zeta^2\right]$
$\doteq \tan^{-1} 2.86(\omega_n t_d)\left\{4.08(\omega_n t_d-1)^2+\sqrt{16.7(\omega_n t_d-1)^4+1}\right\}^{1/2}$	$\tan^{-1}\frac{t_p 2\zeta\omega_n\sqrt{1-\zeta^2}}{\pi\left[\sqrt{4\zeta^4+1}-2\zeta^2\right]^{1/2}}$
$\doteq \left(\frac{1+0.7\zeta}{t_d}\right)^2\left(1-2\zeta^2\right)$	$\frac{1}{t_p^2}\left(2\tau^2-\omega_n^2 t_p^2\right)$
$\doteq \frac{1}{2.86(\omega_n t_d-1)\sqrt{1-2.04(\omega_n t_d-1)^2}}$	$\frac{\omega_n^2 t_p^2}{2\pi(\omega_n^2 t_p^2-\tau^2)^{1/2}}$
$\doteq \left(\frac{1+0.7\zeta}{t_d}\right)^2\left(1-2\zeta^2+\sqrt{2-4\zeta^2+4\zeta^4}\right)$	$\frac{\pi^2}{t_p^2(1-\zeta^2)}\left\{1-2\zeta^2+\sqrt{2-\right.$
1	$\doteq \frac{t_p}{\pi}(1+0.7\zeta)\sqrt{1-\zeta^2}$
$\doteq \frac{\pi t_d}{(1+0.7\zeta)\sqrt{1-\zeta^2}}$	1
$\doteq 1+e^{-\frac{1.427\pi(\omega_n t_d-1)}{1-2.04(\omega_n t_d-1)^2}}$	$1+e^{-\zeta\omega_n t_p}$

t_p	c_p
$\frac{\pi^2}{t_p^2(1-\zeta^2)}\left[\sqrt{4\zeta^4+1}-2\zeta^2\right]$	$\frac{\omega_n^2}{\pi^2+\ln^2(c_p-1)}\left[\sqrt{\pi^4+2\pi^2\ln^2(c_p-1)+5\ln^4(c_p-1)}-2\ln^2\right.$
$\tan^{-1}\frac{t_p 2\zeta\omega_n\sqrt{1-\zeta^2}}{\pi\left[\sqrt{4\zeta^4+1}-2\zeta^2\right]^{1/2}}$	$\tan^{-1}\left[\frac{2\ln(c_p-1)}{\pi^2+\ln^2(c_p-1)}\right]\left\{2\ln^2(c_p-1)+\left[\pi^4+2\pi^2\ln^2(c_p-1)+\right.\right.$
$\frac{1}{t_p^2}(2\pi^2-\omega_n^2t_p^2)$	$\frac{\omega_n^2}{\pi^2+\ln^2(c_p-1)}\left[\pi^2-\ln^2(c_p-1)\right]$
$\frac{\omega_n^2t_p^2}{2\pi(\omega_n^2t_p^2-\pi^2)^{1/2}}$	$\frac{\pi^2+\ln^2(c_p-1)}{2\pi\ln(c_p-1)}$
$\frac{\pi^2}{t_p^2(1-\zeta^2)}\left\{1-2\zeta^2+\sqrt{2-4\zeta^2+4\zeta^4}\right\}$	$\omega_n^2\left[\frac{\pi^2-\ln^2(c_p-1)+\sqrt{2\pi^4+2\ln^4(c_p-1)}}{\pi^2+\ln^2(c_p-1)}\right]$
$\doteq\frac{t_p}{\pi}(1+0.7\zeta)\sqrt{1-\zeta^2}$	$\doteq\frac{1}{\omega_n}\left[1+0.7\frac{\ln(c_p-1)}{\sqrt{\pi^2+\ln^2(c_p-1)}}\right]$
1	$-\frac{\ln(c_p-1)}{\zeta\omega_n}$
$1+e^{-\zeta\omega_n t_p}$	1

	c_p
	$\frac{\omega_n^2}{\pi^2 + \ln^2 (c_p - 1)}\left[\sqrt{\pi^4 + 2\pi^2 \ln^2 (c_p - 1) + 5 \ln^4 (c_p - 1)} - 2 \ln^2 (c_p - 1)\right]$
	$\tan^{-1}\left[\frac{2 \ln (c_p - 1)}{\pi^2 + \ln^2 (c_p - 1)}\right]\left\{2 \ln^2 (c_p - 1) + \left[\pi^4 + 2\pi^2 \ln^2 (c_p - 1) + 5 \ln^4 (c_p - 1)\right]^{1/2}\right\}^{1/2}$
	$\frac{\omega_n^2}{\pi^2 + \ln^2 (c_p - 1)}\left[\pi^2 - \ln^2 (c_p - 1)\right]$
	$\frac{\pi^2 + \ln^2 (c_p - 1)}{2\pi \ln (c_p - 1)}$
$+ 4\zeta^4)$	$\omega_n^2\left[\frac{\pi^2 - \ln^2 (c_p - 1) + \sqrt{2\pi^4 + 2 \ln^4 (c_p - 1)}}{\pi^2 + \ln^2 (c_p - 1)}\right]$
	$\doteq \frac{1}{\omega_n}\left[1 + 0.7 \frac{\ln (c_p - 1)}{\sqrt{\pi^2 + \ln^2 (c_p - 1)}}\right]$
	$-\frac{\ln (c_p - 1)}{\zeta\omega_n}$
	1

3

FORMULAE FOR INDICIAL RESPONSE CHARACTERISTICS

Exact formulae are not available for some of the indicial response characteristics, and the range of validity of the approximations quoted in Table II-B requires some consideration.

The indicial response of a closed-loop system having the open-loop transfer function $G(s) = \frac{\kappa}{s(s + \lambda)}$ is

$$c(t) = 1 + \frac{1}{\sqrt{1 - \zeta^2}} e^{-\zeta\omega_n t} \sin (\omega_n \sqrt{1 - \zeta^2}\, t - \psi) \qquad (8)$$

where $\psi = \tan^{-1} \frac{\sqrt{1 - \zeta^2}}{-\zeta} = \pi - \cos^{-1}\zeta$

$$\omega_n = \sqrt{\kappa}$$

and $\zeta = \frac{\lambda}{2\sqrt{\kappa}}$

This equation can be solved exactly for $\frac{dc(t)}{dt} = 0$, yielding time-to-peak and peak overshoot.

Settling time is commonly approximated by considering only the envelope of the response

$$c_e(t) = 1 + \frac{e^{-\zeta\omega_n t}}{\sqrt{1 - \zeta^2}} \qquad (9)$$

For an overshooting response $c(t_s) = 1.05$. Putting $c_e(t_s) = 1.05$ in Eq 9 yields

$$t_s = \frac{\ln(0.05\sqrt{1-\zeta^2})}{-\zeta\omega_n} \qquad (10)$$

Equation 10 differs from the approximation given on pages 22-41 of Ref. 36 where the $\sqrt{1 - \zeta^2}$ term is replaced by unity. Figure 6 compares Eq 10 and the further approximation $t_s \doteq \frac{3}{\zeta\omega_n}$ given in Ref. 1 with the settling time obtained by direct measurement of analog computer responses. It should be noted that approximations to settling time (e.g., $\frac{3}{\zeta\omega_n}$) do not reproduce the sawtooth shape of the exact graph.

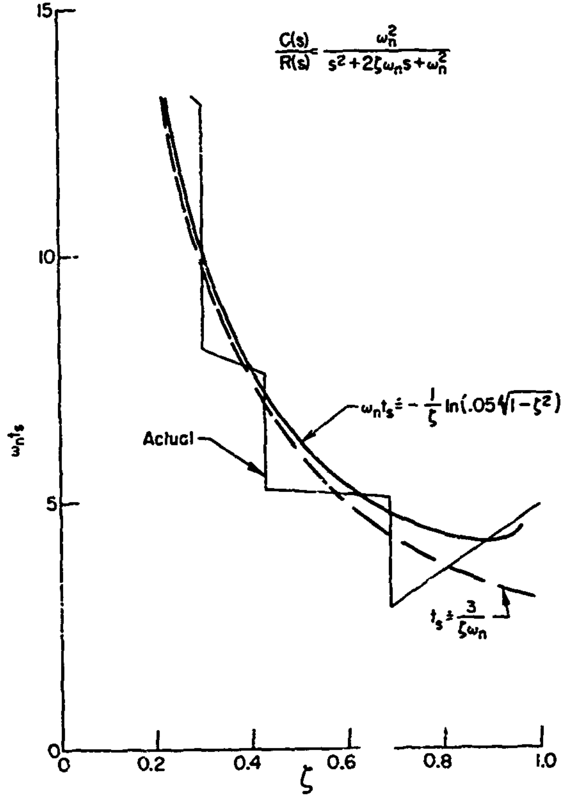

Figure 6. Comparison of Actual and Approximate Settling Times

The Equivalent Time Constant measure is most appropriate for systems with a dominant first-order mode. To the extent that this obtains, $T_{eq} \doteq 1/\omega_c$, and a particularly simple connection exists between the frequency and time domains. When extended to second-order systems, the variation of Equivalent Time Constant with ζ is more regular than the settling time, although the measure is actually inappropriate. For instance, Fig. 7 illustrates measured T_{eq}, and the approximation of Ref. 36 $T_{eq} = 1/\omega_c$ for a second-order system, with the open-loop transfer function of Eq 2. Note that the minimum value of T_{eq} occurs at $\zeta = 0$.

Delay time is defined here as the time for the indicial response to achieve 50 percent of its final value. The exact value is compared with the approximation (given in Ref. 72)

$$t_d = \frac{2\zeta}{\omega_n} \tag{11}$$

in Fig. 8. This approximation is satisfactory only in the "optimal" region of $0.6 < \zeta < 0.9$. An empirical linear relation $t_d = \frac{1 + 0.7\zeta}{\omega_n}$ is more generally applicable. Again, as with T_{eq}, minimum delay time is achieved at $\zeta = 0$. Equation 11 is extended to higher-order systems in Chapter IV, where it is shown that for optimal systems, simple and accurate approximations to delay time can be obtained.

Rise time is defined as the time for the indicial response to rise from 10 to 90 percent of its final value. Reference 36 presents the simple approximation $t_R = \frac{1.5}{\omega_b}$. This is compared with the exact rise time, and a more refined approximation (Eq 12) in Fig. [illegible]

$$t_R = \frac{7.04\zeta^2 + 0.2}{2\zeta\omega_n} \tag{12}$$

A minimum rise time system possesses a low ζ. An approximate formula for rise time is given in Ref. 25

$$t_R^2 \doteq -2\pi\left[2E_2 + E_1^2\right] \tag{13}$$

where E_1 and E_2 are the velocity and acceleration dynamic error coefficients defined at the bottom of Table II-B. For the second order system considered, the accuracy of this approximation is far inferior to Eq 12.

$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

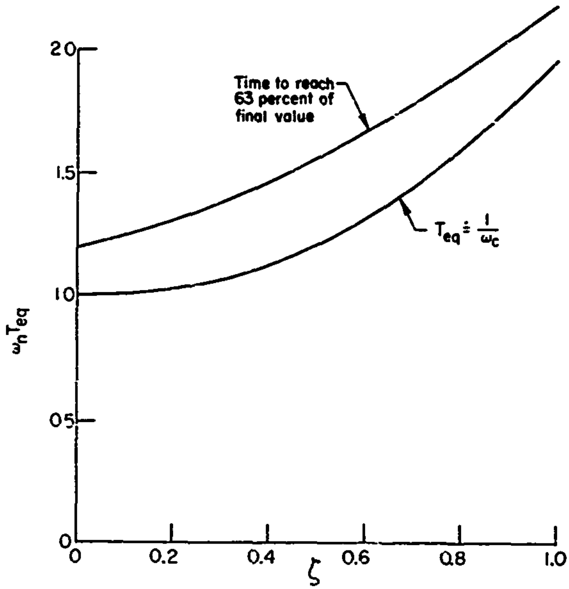

Figure 7. Comparison of Actual and Approximate Equivalent Time Constants

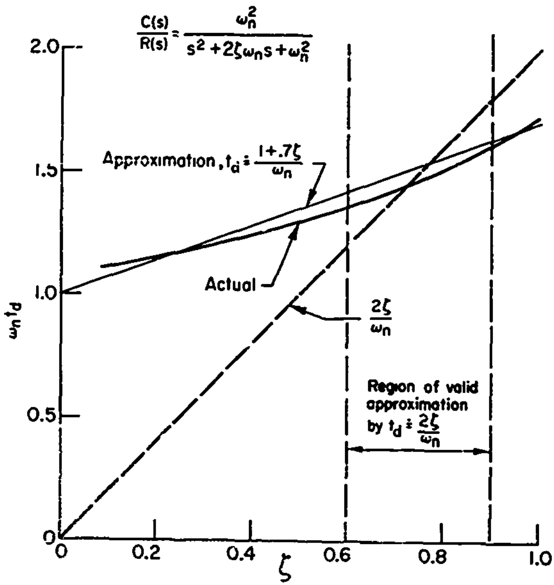

Figure 8. Comparison of Actual and Approximate Delay Times

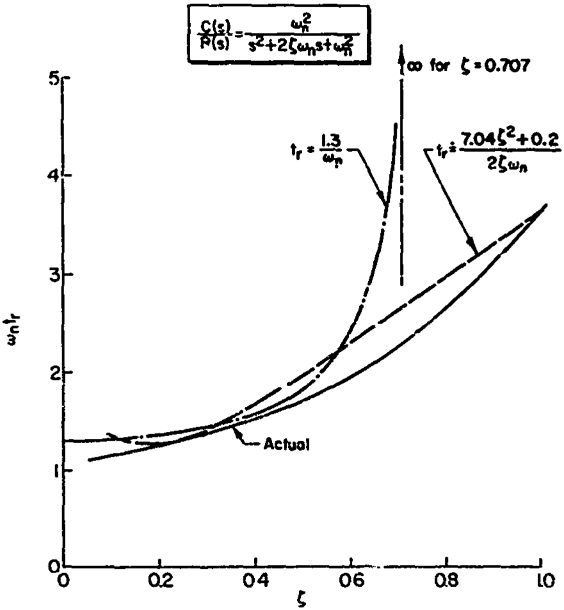

Figure 9. Comparison of Actual and Approximate Rise Times

Figures 10 and 11 illustrate the peak overshoot and time-to-peak, respectively, for a unit-numerator second-order system. The minima occur at $\zeta = 1$ and $\zeta = 0$, respectively.

It should now be apparent that a criterion based on a single indicial response time or transfer function measure is unlikely to be valid for the variety of systems encountered in flight control optimization. Most of the measures specify only particular elements of the response, and/or are restricted to special types of $G(s)$ behavior in the region $|G(j\omega)| \approx 1$. Minimization of the measures results either in low damping ratios, low phase margins, etc., or is unselective. The major use of the indicial response and elementary transfer function measures is for specification purposes; this will be discussed further below.

One indicial response measure, i.e., settling time, is a possible exception to the general statement above. Minimum, or nearly minimum settling time is often used as a criterion, and because it gives a $\zeta = 0.69$ for second-order systems, it is worth examining in the light of the requirements for criteria.

Settling time is a possible exception to the statement above. Since "good" indicial response implies low settling time, it is reasonable to inquire whether the criterion of low settling time always results in indicial responses which are satisfactory in relation to overshoot and the other parameters defining the "shape" of the response. Figure 6 shows that for a unit-numerator normalized second-order system, the criterion of minimum settling time selects $\zeta = 0.69$. However, the sawtooth graph yields almost equal settling times for $0.43 < \zeta < 0.68$, with sudden jumps at each end of this range. Similar discontinuous selectivity characteristics occur with higher-order systems, as shown in Ref. 32 (which presents an exhaustive investigation of settling times for higher-order systems, including standard forms of first- through eighth-order). These discontinuities can be removed by using approximations to settling time (such as $t_s = 3/\zeta\omega_n$, see Fig. 6), but such approximations are difficult to derive for higher-order systems, unless dominant modes can be identified. In general, therefore, minimum settling time cannot be recommended for use as a sole criterion.

Minimization of the product of time-to-peak and peak overshoot, proposed as a criterion in Ref. 24, is also examined in Ref. 32. Its failure to discriminate against undershooting responses and frequent selection of very poor response make this characteristic unsuitable for use as a performance criterion.

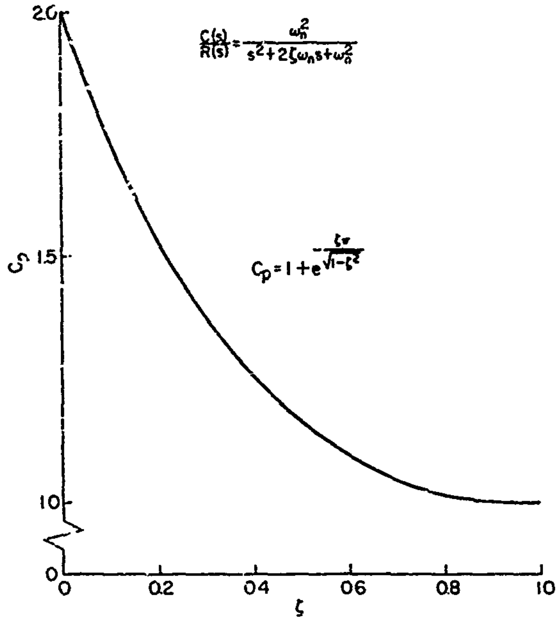

Figure 10. Peak Overshoot of Unit-Numerator Second-Order System

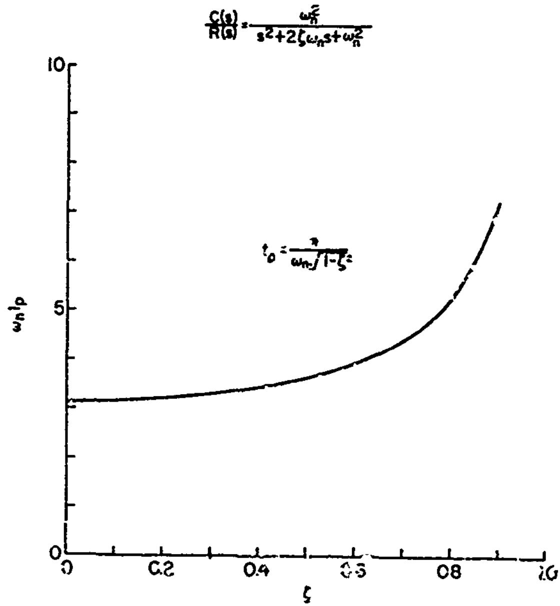

Figure 11. Time-to-Peak of a Unit-Numerator Second-Order System

The use of error coefficients as performance criteria is discussed in Chapter V. It is shown that each error coefficient is proportional to a time-weighted integral of the error response to an impulsive input. The error coefficients are thus the sum of positive and negative error integrals, and fail to indicate whether a small value of this sum represents a small absolute error, or a small difference between large positive and negative errors. It is concluded, therefore, that error coefficients by themselves do not provide valid performance criteria for general systems.

GENERAL CORRELATION OF SYSTEM CHARACTERISTICS

It is not surprising that none of the indicial response characteristics considered in this chapter provides an acceptable performance criterion. These characteristics are suitable for performance specification, rather than optimization. For example, it may be convenient to specify system performance in terms of bandwidth and time-to-peak. It is important that any specification framed in terms of system characteristics should be practically realizable (i.e., mutually exclusive values or limits of characteristics must be avoided). To facilitate this process, it is desirable to have a complete correlation expressing each system characteristic in terms of any other system characteristic. Table III has been prepared for this purpose. Crossover frequency, bandwidth, phase margin, peak frequency, magnification ratio, time-to-peak, peak overshoot, and delay time are all expressed in terms of ζ, ω_n, and each other, for a second-order system.

For flight control systems, despite the presence of numerous unalterable elements associated with the vehicle configuration, the variety of possible systems is so great that no measure of the indicial response based upon any single instant, or measure of transfer function characteristics at a single frequency, can hope to provide more than necessary rather than sufficient conditions for system goodness. Recognition of this fact has led many investigators to propose measures based upon integrated functions of the indicial error response. These indicial error measures are discussed in the next chapter.

The preceding discussion of system characteristics has examined the merit of each characteristic as a performance criterion for unit-numerator second-order systems. This type of system was selected because many equivalent flight control systems are in this category, and consequently it provides a fair basis

for initial assessment. System characteristics that do not yield satisfactory performance criteria for second-order systems need not be considered further in the search for a satisfactory performance criterion. The diversity of possible third- and higher-order systems precludes a general correlation of system characteristics, as has been achieved for second-order systems. However, there is some value in collecting the limited data available on system characteristics for third- and higher-order systems, and comparing the results, wherever practicable.

The well-known charts of Chestnut and Mayer (Ref. 15) reproduced in Ref. 36 correlate time-to-peak and settling time with frequency response characteristics. Elgerd (Ref. 24) presents indicial response time histories of a variety of third-order systems from which rise time, time-to-peak, etc., may be measured directly. To examine the implications of his results, and to compare them with those obtained from other sources, a standardized third-order unit-numerator system is considered.

$$\frac{C(s)}{R(s)} = \frac{(1.13)^2/T}{\left(s + \frac{1}{T}\right)\left[s^2 + 1.13s + (1.13)^2\right]} \tag{14}$$

This form is not covered by the charts of Ref. 36, but is discussed by Clement (Ref. 17), who presents rise time, settling time, and peak overshoot for a normalized system having the following transfer function:

$$\frac{C(s)}{R(s)} = \frac{1}{(1 + \delta s)(s^2 + 2\zeta s + 1)} \tag{15}$$

Figure 12 illustrates Clement's results on settling time for the standardized system of Eq 14. The points marked denote values checked by means of the transient responses of Ref. 24. The agreement is good, although allowance must be made for the fact that the sawtooth shape of the settling time graph has been smoothed in Ref. 17.

Burnett and Shumate (Ref. 13) correlate rise time with peak power for a variety of third-order systems. The resulting values of rise time and overshoot for the system of Eq 14 are presented in Fig. 13, from which it will be seen that the agreement with Ref. 24 is generally close.

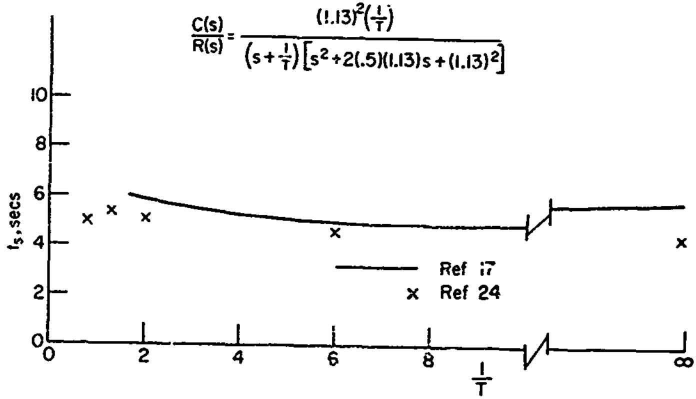

Figure 12. Settling Time of a Third-Order System

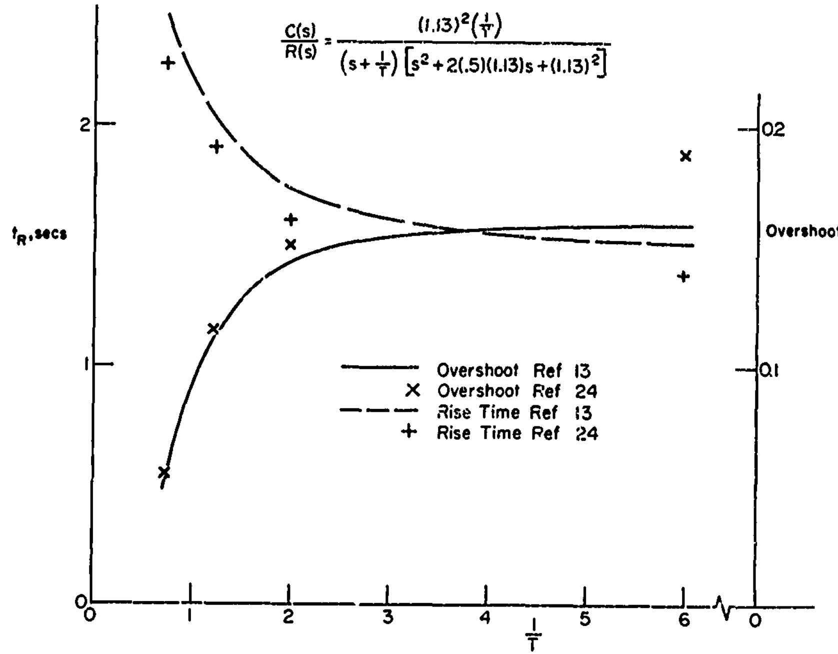

Figure 13. Rise Time and Peak Overshoot of a Third-Order System

CHAPTER III

INDICIAL ERROR MEASURES

The term "indicial error measure" will be used to denote integrated functions of the error response to a step input. Table IV summarizes all the indicial error measures that have been suggested as criteria in the references listed at the end of this report. Most indicial error measures are of the form

$$IU = \int_0^\infty U dt \tag{16}$$

$$ITU = \int_0^\infty tU dt \tag{17}$$

$$IT^2U = \int_0^\infty t^2 U dt \tag{18}$$

etc.,

where U is a general function of error, such as E^2, $|E|$, etc., not involving time explicitly.

These measures are meaningful only for zero-position-error systems; most of the published investigations of the relative merits of these measures have been concerned with their application to systems having unit-numerator closed-loop input-output transfer functions. Since many flight control systems reduce to unit-numerator equivalent systems, this is a fair as well as convenient basis for assessment of criteria. To achieve compact presentation, it is customary to normalize the systems and associated performance measures. Thus, a system having the transfer function

$$\frac{C(\gamma)}{R(\gamma)} = \frac{P_m\gamma^m + \dots + P_2\gamma^2 + P_1\gamma + P_0}{Q_n\gamma^n + \dots + Q_2\gamma^2 + Q_1\gamma + Q_0} \tag{19}$$

where γ is the Laplace transform variable at this point (elsewhere s is employed)

can be put in normalized form by the procedure of Ref. 37, reproduced below for

ease of reference.

1. Define a constant Ω_0 so that

$$\Omega_0^n = \frac{Q_0}{Q_n} \tag{20}$$

2. Define new coefficients for the numerator and denominator terms

$$q_i = \frac{Q_i}{\Omega_0^{n-i}Q_n} \quad , \quad i = 1, 2, \cdots n \tag{21}$$

$$p_i = \frac{P_i}{\Omega_0^{n-i}Q_n} \quad , \quad i = 0, 1, 2, \cdots m \tag{22}$$

3. Divide the numerator and denominator of Eq 19 by Q_n, and apply the definitions of Eq 20, 21, and 22. The transfer function then becomes

$$\frac{C(\gamma)}{R(\gamma)} = \frac{p_m\Omega_0^{n-m}\gamma^m + \cdots + p_2\Omega_0^{n-2}\gamma^2 + p_1\Omega_0^{n-1}\gamma + p_0\Omega_0^n}{\gamma^n + q_{n-1}\Omega_0\gamma^{n-1} + \cdots + q_2\Omega_0^{n-2}\gamma^2 + q_1\Omega_0^{n-1}\gamma + \Omega_0^n} \tag{23}$$

4. Introduce a new transfer variable so that

$$s = \frac{\gamma}{\Omega_0} \tag{24}$$

Then the transfer function reduces finally to the normalized form

$$\frac{C(s)}{R(s)} = \frac{p_m s^m + \cdots + p_2 s^2 + p_1 s + p_0}{s^n + q_{n-1}s^{n-1} + \cdots + q_2 s^2 + q_1 s + 1} \tag{25}$$

From dimensional considerations it can be shown that to convert a normalized performance measure to its denormalized form, the following relations should be used:

$$IU_{(\Omega_0 \neq 1)} = \frac{1}{\Omega_0} IU_{(\Omega_0 = 1)} \tag{26}$$

$$ITU_{(\Omega_0 \neq 1)} = \frac{1}{\Omega_0^2} ITU_{(\Omega_0 = 1)} \tag{27}$$

$$IT^2U_{(\Omega_0 \neq 1)} = \frac{1}{\Omega_0^3} IT^2U_{(\Omega_0 = 1)} \tag{28}$$

The normalized form of the closed-loop second-order system considered previously

TABLE IV

INDICIAL ERROR MEAS

SYMBOL	MEASURE	ASSOCIATED CRITERION	CRITERION A (BASED UPON UNIT-D
IE	Control area [61] $\int_0^\infty e(t)dt$	Minimum value [37,51,59,62,71]	If constrained to nonoversho $\zeta = 1.0$. If overshoots are selected. Lack of validity as a criterion. (Ref. 37)
ITE	Time weighted control area $\int_0^\infty te(t)dt$	Minimum value [59]	If constrained to nonoversho $\zeta = 1.0$. Applies heavier we selectivity) to system with area criterion. If overshoo selected. Lack of general v criterion (Ref. 37)
IT^{n}E	$\int_0^\infty t^n e(t)dt$	Minimum value	Similar to $\int_0^\infty te(t)dt$, ex $\zeta > 1$ even more.
IAE	$\int_0^\infty \lvert e(t)\rvert dt$	Minimum value [14,30]	For normalized second-order $\zeta = 0.63$. Criterion is mode low-order unit numerator sys zero-velocity error systems Easily mechanized on analog selectivity eliminates as a
ITAE	$\int_0^\infty t\lvert e(t)\rvert dt$	Minimum value [37]	Selective, reliable, and eas systems (through eighth-orde Approaches the ideal criteri optimization employing analo
IT^{2}AE	$\int_0^\infty t^2\lvert e(t)\rvert dt$	Minimum value [37]	Highly selective, giving ζ = second-order unit-numerator is very complicated and less also less convenient than IT mechanization.
IE^2	$\int_0^\infty e^2(t)dt$	Minimum value [41]	Selects a value of ζ = 0.5 fo Lack of selectivity and spec oscillatory responses for hi makes use as a criterion uns

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TABLE IV

INDICIAL ERROR MEASURES

ION	CRITERION ASSESSMENT (BASED UPON UNIT-NUMERATOR SYSTEMS)	ASSOCIATED APPLICATION TECHNIQUES AND AIDS; REMARKS
,62,71	If constrained to nonovershooting second-order system, $\zeta = 1.0$. If overshoots are allowed, $\zeta = 0$ is selected. Lack of validity and selectivity eliminates as a criterion. (Ref. 37)	
	If constrained to nonovershooting second-order system, $\zeta = 1.0$. Applies heavier weighting (and more selectivity) to system with $\zeta > 1$ than the control area criterion. If overshoots are allowed, $\zeta = 0$ is selected. Lack of general validity eliminates as a criterion. (Ref. 37)	For impulsive response (not step) each of these measures is proportional to a particular error coefficient. (See Table II-B and Chapter V)
	Similar to $\int_0^\infty te(t)dt$, except weights responses for $\zeta > 1$ even more.	
	For normalized second-order system, criterion selects $\zeta = 0.63$. Criterion is moderately selective on low-order unit numerator system, but nonselective on zero-velocity error systems and higher-order systems. Easily mechanized on analog computer. Lack of selectivity eliminates as a criterion. (Ref. 37)	Analytic form given in Chapter III (Eq 55) and Appendix A.
	Selective, reliable, and easy to apply for all systems (through eighth-order) investigated. Approaches the ideal criterion for routine optimization employing analog computers. (Ref. 37)	Criterion is thoroughly explored and is good in nearly every respect, except complicated nature of analytic forms. Analog computer-derived standard forms and transient responses exist for zero-displacement error systems through eighth-order, zero-velocity and zero-acceleration error systems through sixth-order. (Ref. 37)
	Highly selective, giving $\zeta = 0.6$ for a normalized second-order unit-numerator system. Analytic form is very complicated and less easy to apply than ITAE; also less convenient than ITAE for analog computer mechanization.	Criterion is much less thoroughly explored than ITAE. Possibly not worth the extra complication over ITAE.
	Selects a value of $\zeta = 0.5$ for second-order systems. Lack of selectivity and specification of highly oscillatory responses for higher-order systems makes use as a criterion unsuitable. (Ref. 37)	Is the simplest of the higher-order measures to apply analytically (Ref. 63), viz: $\int_0^\infty e^2(t)dt = \frac{1}{2\pi j}\int_{-j\infty}^{j\infty} E(-s)E(s)ds$ which is a well-tabulated integral. (Ref. 44, 58) General and optimal (standard) forms and transient responses through fourth-order exist (Tables V and VI) and allow a fairly simple determination of effects of uncertainties in system parameters. (Can be evaluated fairly easily by operations on normal design charts.) These two advantages make its use as a criterion common in spite of its relative lack of selectivity and validity.

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TABLE IV
(CONTINUED)

SYMBOL	MEASURE	ASSOCIATED CRITERION	CRITERION ASSESSMENT (BASED UPON UNIT-NUMERATOR SYSTEMS)
ITE^2	$\int_0^\infty te^2(t)dt$	Minimum value[70,72]	Selects a $\zeta = 0.594$ for second-order system. (Ref. 37) Has fair selectivity and reliability for higher-order systems (through fourth order). (Ref. 70) Appears very promising as an ideal unitary criterion in analytic studies.
IT^nE^2	$\int_0^\infty t^n e^2(t)dt$	Minimum value[70,72]	For second-order systems, $\zeta \doteq 0.65$ for $n = 2$. (Ref. 37) Selectivity becomes greater as n increases, and criterion also becomes more complex to assess.
	$\int_0^\infty e^2(t)dt + \alpha \int_0^\infty \left[e^2(t)dt\right]^2 \int_0^\infty \left[\frac{de(t)}{dt}\right]^2 dt$	Minimum value	Yields $\zeta = 0.867$ for second-order system for $\alpha \rightarrow \infty$. (Ref. 73) Has not yet been evaluated for higher order systems.
	$\int_0^\infty \left\{\sum_{n=0}^{N} a_n \left[\frac{d^n e(t)}{dt^n}\right]^2\right\} dt$	Minimum $\int_0^\infty \sum_{n=0}^{N} a_n \left(\frac{d^n e(t)}{dt^n}\right)^2 dt$ while another parameter, like $\int_0^\infty (\text{torque})^2 dt = \text{const.}$	Fails the ready applicability test for the constant coefficient systems considered here. Choice of a_n is arbitrary, making the general form of this measure difficult to assess in the present terms of reference.
	$\int_0^\infty F[e, t, \lambda_i]\mu(t)dt$	Minimum value[80,81]	Probably fails ready applicability test. Choice of F is arbitrary, making the general form of this measure ill-defined and difficult to assess.
	$\int_0^\infty [c(t) - c(\infty)\mu(t-T_0)]^2 dt$ where $\mu(t)$ is a unit step and T_0 is a time delay	Minimum value[5]	ASSOCIATED RESPONSE MEASURES Not yet evaluated. A logical form for systems with finite position error. Choice of T_0 is arbitrary.
	$\int_0^\infty [r(t - T_0) - c(t)]^2 dt$	Minimum value[66,69]	Not yet evaluated. Lack of fixed value for T_0 makes this measure vague and indefinite.
	$\int_0^\infty c(t)dt$ (impulse input) $\int_0^\infty t^n c(t)dt$ (impulse input)	$N - \int_0^\infty c(t)dt$[5] $N - \int_0^\infty t^n c(t)dt$ where N is a predetermined constant	Unsatisfactory response for higher order systems. (Ref. 34)

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TABLE IV
(CONTINUED)

ASSOCIATED CRITERION	CRITERION ASSESSMENT (BASED UPON UNIT-NUMERATOR SYSTEMS)	ASSOCIATED APPLICATION TECHNIQUES AND ALL. REMARKS
...nimum value [0,7]	Selects a $\zeta = 0.594$ for second-order system. (Ref 57) Has fair sele- vity and reliability for higher-order systems (through fourth order). (Ref. 70) Appears very promising as an ideal unitary criterion in analytic studies.	Relatively easy to apply analytically as it can be evaluated using same integral form as noted above, viz: $\int_0^\infty te^2(t)dt = -\lim_{\sigma \to 0} \frac{\partial}{\partial\sigma}\left\{\frac{1}{4\pi j}\int_{\sigma-j\infty}^{\sigma+j\infty} E(\sigma+s)E(\sigma-s)ds\right\}$ where σ is carried as a parameter in the integration. General and optimal (standard) forms and transient responses through fourth order exist (Tables 5 and 6) and allow analytic evaluation of effects of system uncertainties.
...nimum value [70,71]	For second-order systems, $\zeta \doteq 0.65$ for $n = 2$. (Ref. 57) Selectivity becomes greater as n increases, and criterion also becomes more complex to assess.	Analytical expression becomes more difficult as n increases, viz: $\int_0^\infty t^n e^2(t)dt = (-1)^n \lim_{\sigma \to 0} \frac{\partial^n}{\partial\sigma^n}\left\{\frac{1}{4\pi j}\int_{\sigma-j\infty}^{\sigma+j\infty} E(\sigma+s)E(\sigma-s)ds\right\}$ Standard forms and associated transient responses have been evolved for n = 1, 2, 3 using computers. (Ref. 70) General (i.e., nonminimized) forms presented in Table 5. Standard forms in Table 6. Generally not worth the extra computing effort over $\int_0^\infty te^2(t)dt$.
...t Minimum value	Yields $\zeta = 0.867$ for second-order system for $\alpha \to \infty$. (Ref. 73) Has not yet been evaluated for higher order systems.	Extremum value can be found analytically (Ref. 73), although even simple forms are fairly complex. Selection of α is arbitrary.
...nimum $\int_0^\infty \sum_{n=0}^{N} a_n \left(\frac{d^n e(t)}{dt^n}\right)^2 dt$ ...le another parameter, like ...$\int^\infty$ $(\text{torque})^2 dt$ = const.	Fails the ready applicability test for the constant coefficient systems considered here. Choice of a_n is arbitrary, making the general form of this measure difficult to assess in the present terms of reference.	A typical criterion form suitable for solution via dynamic programming techniques. This one has been selected from a large number of relatively unevaluated "generalizations" proposed recently (Ref. 7, 8, 9, 31, 46, 47) because it reduces in limiting cases to criteria that have merit for simple constant coefficient systems, e.g., minimum $\int_0^\infty e^2 dt$ while $\int_0^\infty (\text{torque})^2 dt = \text{const}$ This selects $\zeta = 0.7$ for a second-order system. (Ref. 63) Has not been evaluated for higher-order systems. Will become more familiar and important as dynamic programming applications expand.
...nimum value [80,81]	Probably fails ready applicability test. Choice of F is arbitrary, making the general form of this measure ill-defined and difficult to assess.	$F[e, t, \lambda_i]$ is a general functional of error, time, and system parameters, λ_i. $p(t)$ is the probability that the output will be used. Proposed as an all-encompassing criterion.
...nimum value [3]	ASSOCIATED RESPONSE MEASURES Not yet evaluated. A logical form for systems with finite position error. Choice of T_0 is arbitrary.	A clever use of Laguerre functions in evaluating the integral appears in Ref 3, which also considers the case with an impulse input. May have application to special control problems.
...nimum value [66,69]	Not yet evaluated. Lack of fixed value for T_0 makes this measure vague and indefinite.	Proposed as a generalized performance measure for transient inputs. Reduces when $T_0 = 0$ to $\int_0^\infty e^2(t)dt$.
$\int_0^\infty c(t)dt$ [5] $\int_0^\infty t^n c(t)dt$ ...re N is a predetermined ...stant	Unsatisfactory response for higher order systems. (Ref. 34)	Simply related to input-output closed-loop transfer function in same way as error coefficients are related to time-weighted impulse response of error (see Chapter V).

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is

$$\frac{C(s)}{R(s)} = \frac{1}{s^2 + 2\zeta s + 1} \tag{29}$$

For systems with a nonzero steady-state error following a step input, indicial error measures are not applicable, as noted above. However, it is easy to derive associated integrated response measures yielding finite criteria values; four such measures are appended to Table IV.

The measures will now be examined with regard to their validity, selectivity, and ease of application.

As noted in Table IV, IE, ITE, and IT^nE are all invalidated by their inability to discriminate between responses which are good in the sense that the absolute error is small or decays rapidly, and oscillatory responses which are lightly damped, and in which the large positive and negative errors approximately cancel. Similar drawbacks apply to the associated impulsive response measures $\int_0^\infty c(t)dt$, $\int_0^\infty t^n c(t)dt$. Therefore, these measures will not be discussed further in this chapter, except insofar as they represent limiting values of other measures (e.g., ITAE = IE for nonovershooting responses). Most of the remainder of this chapter will be devoted to a study of the measures IE^2, ITE^2, IT^2E , IAE, ITAE, and IT^2AE. Particular attention will be given to the analytic evaluation of these measures, not only because of the insight that analytic expressions provide, but also because they enable a check to be made on published values obtained by mechanization of analog computer responses. As will be shown, several errors have been detected.

DERIVATION OF IE^2, ITE^2, AND IT^2E^2

Tables giving IE^2 for nonunit numerator systems of first- through seventh-order in terms of numerator and denominator coefficients are given in Ref. 44. Appendix E of Ref. 58 extends these tables through tenth-order, and corrects an error in the seventh-order integral of Ref. 44. These literal expressions are lengthy, and Ref. 44 notes that the integrals can be expressed more compactly in terms of Hurwitz determinants, a point which is discussed further in Ref. 6 and 45. Tables for ITE^2 (and IE^2) are given by Westcott (Ref. 79) for systems of first- through fourth-order. The topic also been studied by Knothe (Ref. 48) and Stone (Ref. 70), using a somewhat different approach than the previous

references. Stone's procedure, and that of Westcott, is outlined very briefly below.

The Laplace transform of the error can be expressed as

$$E(s) = \frac{d_0 s^{n-1} + d_1 s^{n-1} + \cdots + d_{n-1}}{a_0 s^n + a_1 s^{n-1} + \cdots + a_n} \tag{30}$$

Stone first calculates literal expressions for

$$\mathcal{L}\left[\mathcal{L}^{-1} \frac{d_0 s^{n-1} + d_1 s^{n-1} + \cdots + d_{n-1}}{a_0 s^n + a_1 s^{n-1} + \cdots + a_n}\right]^2 \tag{31}$$

IE^2 may then be obtained by use of the final value theorem

$$IE^2 = \lim_{s \to 0} s \cdot \frac{1}{s} \cdot \mathcal{L}\left[\mathcal{L}^{-1} E(s)\right]^2 = \lim_{s \to 0} s \cdot \frac{1}{s} \mathcal{L}\left[\mathcal{L}^{-1} \frac{d_0 s^{n-1} + d_1 s^{n-1} + \cdots + d_{n-1}}{a_0 s^n + a_1 s^{n-1} + \cdots + a_n}\right]^2 \tag{32}$$

$$ITE^2 = \lim_{s \to 0} s \cdot \frac{1}{s} \cdot \frac{-d}{ds} \mathcal{L}\left[\mathcal{L}^{-1} E(s)\right]^2 \tag{33}$$

$$IT^2E^2 = \lim_{s \to 0} s \cdot \frac{1}{s} \cdot \frac{d^2}{ds^2} \mathcal{L}\left[\mathcal{L}^{-1} E(s)\right]^2 \tag{34}$$

The procedure is simple, in principle, once the literal form for Eq 31 has been obtained. However, in practice, for systems above the third-order this literal form becomes exceedingly long. (Stone presents such a form for a nonunit numerator fourth-order system which occupies nine pages.) Some simplification may be introduced into the differentiation required to obtain IT^2E^2 by the a priori neglect of terms in s^3 and above, which vanish when s is allowed to tend to zero in the second derivative. Nevertheless, Westcott's procedure seems to be somewhat briefer, and is preferred for the purpose of obtaining ITE^2 and IT^2E^2.

Westcott, following Ref. 44 and 58, employs Parseval's theorem to express IE^2 as

$$IE^2 = \int_0^{\infty} \left[e(t)\right]^2 dt = \lim_{\sigma \to 0} \frac{1}{2\pi j} \int_{c-j\infty}^{c+j\infty} E(s)E(\sigma - s)ds \tag{35}$$

which, for a stable system, becomes

$$IE^2 = \frac{1}{2\pi j} \int_{-j\infty}^{+j\infty} E(s)E(-s)ds \tag{36}$$

When $E(s)$ is, in addition, a ratio of rational polynomials, Eq 36 is a symmetrical rational function of the poles of $E(s)$, being the sum of the residues at these poles, and can therefore be expressed in terms of the coefficients of $E(s)$ only. The procedure is well-illustrated by considering the simplest case:

$$E(s) = \frac{d_0}{a_0 s + a_1} \tag{37}$$

$$IE^2 = \frac{1}{2\pi j} \int_{-j\infty}^{j\infty} \frac{d_0^2}{(a_0 s + a_1)(-a_0 s + a_1)} ds \tag{38}$$

which has a single pole at $s = -a_1/a_0$ enclosed by the contour integration which is an arbitrarily large semicircle in the left-half plane. The residue at this pole is equal to IE^2, and is given by

$$IE^2 = \lim_{s \to -\frac{a_1}{a_0}} \left[\frac{a_0 s + a_1}{a_0} \cdot \frac{d_0^2}{(a_0 s + a_1)(-a_0 s + a_1)}\right] = \frac{d_0^2}{2a_0 a_1} \tag{39}$$

The procedure for obtaining ITE^2 is indicated by Eq 33.

$$ITE^2 = \int_0^{\infty} t\left[e(t)\right]^2 dt = -\lim_{\sigma_1 \to 0} \frac{\partial}{\partial \sigma_1} \frac{1}{2\pi j} \int_{c-j\infty}^{c+j\infty} E(s)E(\sigma_1 - s)ds \tag{40}$$

It is not possible to take the $\sigma_1 \to 0$ limiting process under the integral sign without first performing the differentiation. To obtain a symmetrical function,

σ_1 assumed to be real and positive, the contour is taken as the midline of the displacement of the E(s) and E(-s) functions, completed by a large semicircle occupying the left half, and part of the right half, complex plane. This is equivalent to simply changing the variable s to $s + (\sigma_1/2)$, and treating the contour integral as an ordinary integral. Equation 36 then becomes

$$ITE^2 = - \lim_{\sigma_1 \to 0} \frac{\partial}{\partial \sigma_1} \frac{1}{2\pi j} \int_{\frac{\sigma_1}{2} - j\infty}^{\frac{\sigma_1}{2} + j\infty} E\left(s + \frac{\sigma_1}{2}\right) E\left(\frac{\sigma_1}{2} - s\right) ds \tag{41}$$

By substituting $\sigma = \sigma_1/2$,

$$ITE^2 = - \lim_{\sigma \to 0} \frac{\partial}{\partial \sigma} \left[\frac{1}{4\pi j} \int_{\sigma - j\infty}^{\sigma + j\infty} E(s + \sigma)E(\sigma - s) ds \right]$$

This may now be solved by making use of the IE^2 tables. For example, taking the simplest case,

$$E(s) = \frac{d_0}{a_0 s + a_1} \tag{42}$$

the integrand becomes

$$\frac{d_0^2}{(a_0 s + a_0\sigma + a_1)(-a_0 s + a_0\sigma + a_1)}$$

The group $a_1 + a_0\sigma$ now corresponds to a_1 in Eq 38; making the appropriate substitution in Eq 40 yields

$$ITE^2 = -\frac{1}{2} \lim_{\sigma \to 0} \frac{\partial}{\partial \sigma} \left(\frac{d_0^2}{2a_0(a_0\sigma + a_1)} \right) = \frac{d_0^2}{4a_1^2} \tag{43}$$

By means of this procedure, Westcott obtains literal forms for ITE^2. The process has been extended by a further $-\partial/\partial\sigma$ operation to obtain IT^2E^2 in the present report.

Table V presents the resulting literal forms for IE^2, ITE^2, and IT^2E^2 for systems of first- through fourth-order. The ITE^2 results have been rederived independently of Westcott's work, and check his values exactly. No general

TABLE V

LITERAL FORMS OF INTEGRALS REQUIRED FOR IE^2, ITE^2 AND IT^{2}E^2 FOR SYSTEMS OF FIRST- THROUGH FOURTH-ORDER

$$s_n = \int_0^\infty \left[\mathcal{L}^{-1} \frac{c_0 s^{n-1} + c_1 s^{n-2} + \dots c_{n-1}}{a_0 s^n + a_1 s^{n-1} + \dots a_n} \right]^2 dt$$

$$s_1 = \frac{c_0^2}{2a_0a_1}$$

$$s_2 = \frac{c_0^2 + \frac{a_0}{a_2} c_1^2}{2a_0a_1}$$

$$s_3 = \frac{a_2 c_0^2 + a_0(c_1^2 - 2c_0c_2) + \frac{a_0a_1}{a_3} c_2^2}{2a_0(a_1a_2 - a_0a_3)}$$

$$s_4 = \frac{c_0^2(a_2a_3 - a_1a_4) + a_0a_3(c_1^2 - 2c_0c_2) + a_0a_1(c_2^2 - 2c_1c_3) + \frac{a_0}{a_4} c_3^2 (a_1a_2 - a_0a_3)}{2a_0(a_1a_2a_3 - a_0a_3^2 - a_1^2a_4)}$$

$$v_n = \int_0^\infty \left[\mathcal{L}^{-1} \frac{c_0 s^{n-1} + c_1 s^{n-2} + \dots c_{n-1}}{a_0 s^n + a_1 s^{n-1} + \dots a_n} \right]^2 t\, dt$$

$$v_1 = \frac{c_0^2}{4a_1^2}$$

$$v_2 = \frac{c_1^2}{4a_2^2} + \frac{1}{2a_1^2}\left(c_0^2 + \frac{a_0}{a_2} c_1^2 - \frac{a_1}{a_2} c_0c_1\right)$$

$$v_3 = \frac{c_2^2}{4a_3^2} - \frac{c_0c_1 + \frac{a_1}{a_3} c_1c_2}{a_1a_2 - a_0a_3} + \frac{\left[c_0^2 + (c_1^2 - 2c_0c_2)(a_0a_2 + a_1^2) + \frac{c_2^2}{a_3}(a_0^2a_3 + a_1^3)\right]}{2(a_1a_2 - a_0a_3)^2}$$

$$v_4 = \frac{c_3^2}{4a_4^2} - \left[\frac{2a_4c_0^2 + 2a_0(c_2^2 - 2c_1c_3) + a_2(c_1^2 - 2c_0c_2) + a_1(c_1c_2 - 3c_0c_3) + a_3c_0c_1}{2(a_1a_2a_3 - a_0a_3^2 - a_1^2a_4)}\right.$$

$$\left. + \frac{\frac{c_2c_3}{a_4}(a_1a_2 - a_0a_3) + \frac{c_3^2}{a_4}(a_1^2 + a_0a_2)}{2(a_1a_2a_3 - a_0a_3^2 - a_1^2a_4)}\right]$$

$$+ \frac{(a_2^2 + a_1a_3 - 4a_0a_4)\left[c_0^2a_3^2 + a_0a_3(c_1^2 - 2c_0c_2) + a_0^2(c_2^2 - 2c_1c_3) - \frac{a_1}{a_4} c_3^2(a_1a_2 - a_0a_3)\right]}{2(a_1a_2a_3 - a_0a_3^2 - a_1^2a_4)^2}$$

TABLE V (Continued)

$$I_n = \int_0^\infty \left[\mathcal{L}^{-1} \frac{d_0 s^{n-1} + d_1 s^{n-2} + \cdots d_{n-1}}{a_0 s^n + a_1 s^{n-1} + \cdots a_n} \right]^2 t^2 dt$$

$$I_1 = \frac{a_0 d_0^2}{a_1^3}$$

$$I_2 = \frac{1}{2 a_1 a_2} \left[d_0^2 + \frac{a_0 a_1 d_1^2 - 2 a_1^2 d_0 d_1 - 4 a_0 a_2 d_0 d_1}{a_1 a_2} + \frac{a_1^4 d_1^2 + 4 a_0 a_2^3 d_0^2 + 4 a_0^2 a_2^2 d_1^2}{a_1^2 a_2^2} \right]$$

$$I_n = \frac{1}{2\Delta_n} \left\{ \ddot{B}_n - \frac{B_n}{\Delta_n} \ddot{\Delta}_n - 2 \frac{\dot{\Delta}_n}{\Delta_n} \left(\dot{B}_n - \frac{B_n}{\Delta_n} \dot{\Delta}_n \right) \right\}$$

where

$$\Delta_3 = 2 a_0 a_3 (a_1 a_2 - a_0 a_3)$$

$$\dot{\Delta}_3 = 2 a_0 (a_0 a_2 a_3 + 2 a_1^2 a_3 + a_1 a_2^2)$$

$$\ddot{\Delta}_3 = 4 a_0 (7 a_0 a_1 a_3 + 2 a_0 a_2^2 + 3 a_1^2 a_2)$$

I_3

$$B_3 = a_2 a_3 d_0^2 + a_0 a_3 (d_1^2 - 2 d_0 d_2) + a_0 a_1 d_2^2$$

$$\dot{B}_3 = d_0^2 (a_2^2 + 2 a_1 a_3) + a_0 a_2 (d_1^2 - 2 d_0 d_2) + a_0 (2 a_1 d_1 d_2 + 3 a_0 d_2^2 + 2 a_3 d_0 d_1)$$

$$\ddot{B}_3 = 2 d_0^2 (3 a_1 a_2 + 3 a_0 a_3) + 4 a_0 d_1 (3 a_0 d_2 + a_1 d_1 + a_2 d_0)$$

$$\Delta_4 = 2 a_0 (a_1 a_2 a_3 - a_0 a_3^2 - a_1^2 a_4)$$

$$\dot{\Delta}_4 = 4 a_0 a_1 (a_2^2 + a_1 a_3 - 4 a_0 a_4)$$

$$\ddot{\Delta}_4 = 16 a_0 (5 a_1^2 a_2 + a_0 a_2^2 + a_0 a_1 a_3 - 4 a_0^2 a_4)$$

I_4

$$B_4 = d_0^2 (a_2 a_3 - a_1 a_4) + a_0 a_3 (d_1^2 - 2 d_0 d_2) + a_0 a_1 (d_2^2 - 2 d_1 d_3) + \frac{a_0}{a_4} d_3^2 (a_1 a_2 - a_0 a_3)$$

$$\dot{B}_4 = 2 d_0^2 (a_2^2 + a_1 a_3 - 2 a_0 a_4) + a_0^2 \left[4 d_2^2 - 8 d_1 d_3 - 2 \frac{a_3}{a_4} d_2 d_3 + 2 \frac{a_2}{a_4} d_3^2 + \frac{a_3^2}{a_4^2} d_3^2 \right]$$
$$+ a_0 a_1 \left[2 d_1 d_2 - 6 d_0 d_3 + 2 \frac{a_2}{a_4} d_2 d_3 + d_3^2 \left(3 \frac{a_1}{a_4} - \frac{a_2 a_3}{a_4^2} \right) \right] + 2 a_0 a_2 (d_1^2 - 2 d_0 d_2) + 2 a_0 a_3 d_0 d_1$$

$$\ddot{B}_4 = 2 d_0^2 (8 a_1 a_2 + 24 a_0 a_3) + a_0^2 \left[16 (d_1 d_2 - 3 d_0 d_3) - 4 \frac{a_3}{a_4} (d_2^2 + 2 d_1 d_3) \right.$$
$$\left. + 4 d_2 d_3 \left(2 \frac{a_2}{a_4} + \frac{a_3^2}{a_4^2} \right) + 2 d_3^2 \left(15 \frac{a_1}{a_4} - \frac{a_2 a_3}{a_4^2} - \frac{a_3^3}{a_4^3} \right) \right]$$
$$+ a_0 a_1 \left[10 d_1^2 - 12 d_0 d_2 + 2 \frac{a_2}{a_4} (d_2^2 + 2 d_1 d_3) + 4 d_2 d_3 \left(3 \frac{a_1}{a_4} - \frac{a_2 a_3}{a_4^2} \right) - 2 d_3^2 \left(3 \frac{a_1 a_3}{a_4^2} - \frac{a_2^2}{a_4^2} - \frac{a_2 a_3^2}{a_4^3} \right) \right]$$
$$+ 8 a_0 a_2 d_0 d_1$$

literal values of IT^2E^2 have previously been published; hence, it is not possible to have a complete independent check on this measure, although, as will be noted in the following section, a partial check does exist.

EVALUATION OF IE^2, ITE^2, AND IT^2E^2 AS PERFORMANCE MEASURES

Indicial responses of normalized unit-numerator minimum IE^2, ITE^2, IT^2E^2, and IT^3E^2 systems are given in Ref. 70. These are reproduced in Fig. 15 for ease of reference, and corresponding closed-loop Bode diagrams are given in Fig. 14. The transfer function coefficients that minimize these normalized IE^2, ITE^2, etc., performance measures are called standard forms, i.e., the partial derivative of the normalized performance measure with respect to each coefficient of the closed-loop transfer function is zero when the transfer function has the appropriate standard form. Table VI presents such standard forms for a variety of performance criteria, including minimum IE^2, ITE^2, and IT^2E^2. These have been obtained by differentiating the general analytic expressions of Table V with respect to the transfer function coefficients. The results agree with those of Ref. 70 which were obtained by a digital computer programmed for iterative minimization. (This provides a partial check on the accuracy of the IT^2E^2 general forms.) Reference 70 also presents standard forms and indicial responses for optimal IT^3E^2 systems. These have not been checked, but are included in Table VI to ensure completeness. The analytic expressions for IT^3E^2 for systems of higher than second-order are very complicated, and IT^3E^2 does not appear to offer advantages over ITE^2 or IT^2E^2 that would be sufficient compensation for the computing efforts involved in practical optimization calculations.

Table VI also lists the open-loop transfer function coefficients associated with the standard forms. As with the closed-loop coefficients, the sensitivity of the normalized performance measure to small changes in these coefficients about the specified values is zero. Note, however, that the unnormalized performance measure is affected by small changes in open-loop coefficients from the standard forms.

The closed-loop $(j\omega)$ Bode diagrams for the IE^2, ITE^2, IT^2E^2, and IT^3E^2 standard forms are given in Fig. 14. (ζ templates correct to one significant figure were used to construct these diagrams.) By operations on a Nichols chart, or by direct factorization of the lower-order systems, the associated open-loop characteristics of phase margin, crossover frequency, etc., have been found.

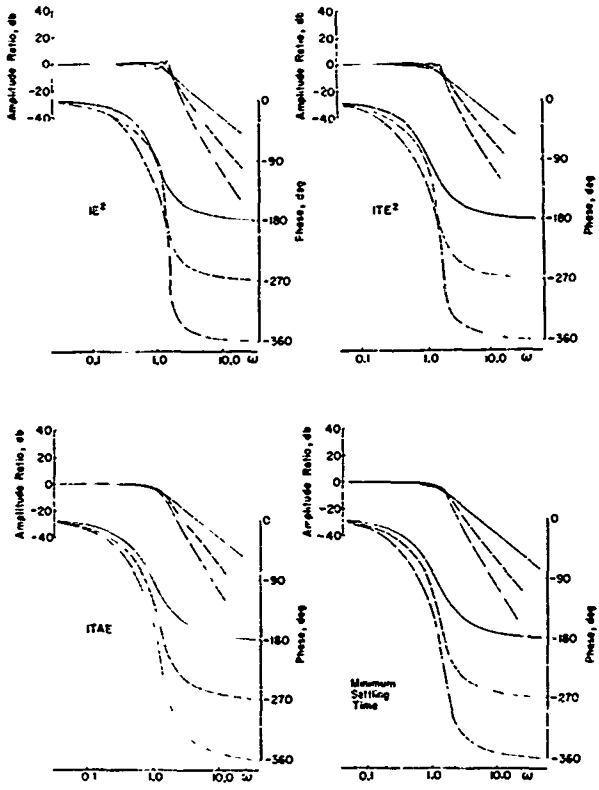

Figure 14. Closed-Loop Bode (jω) Diagrams of Standard Forms

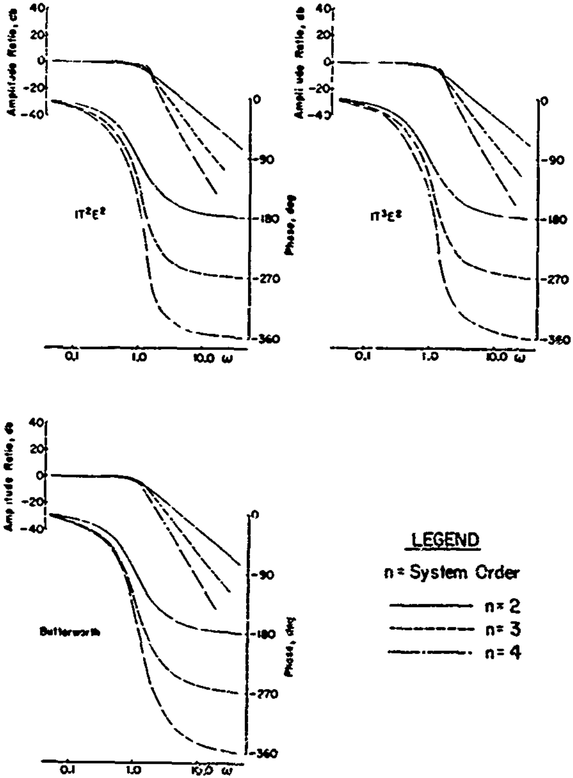

Figure 14. (Continued)

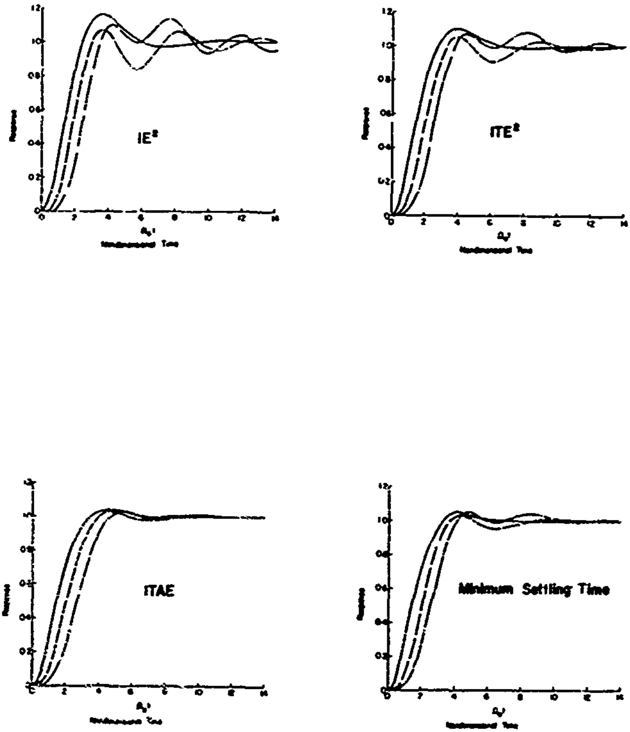

Figure 15. Indicial Responses of Standard Forms

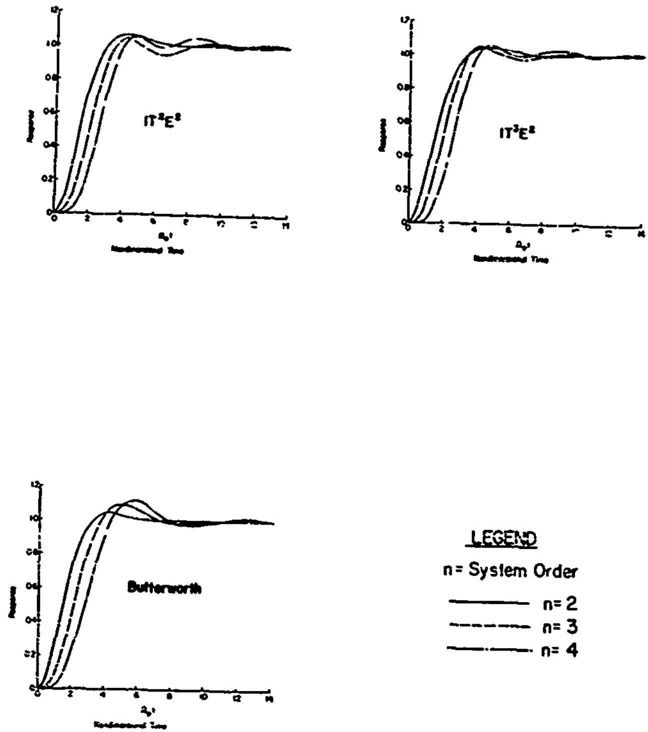

Figure 15. (Continued)

These are presented in Table VII, together with the corresponding values for minimum ITAE systems and other standard forms. Table VII and Fig. 15 indicate that the standard forms having the smoothest indicial responses possess phase margins between 60 and 70 degrees.

For a normalized second-order unit-numerator system having a transfer function described by Eq 29, $\frac{C(s)}{R(s)} = \frac{1}{s^2 + 2\zeta s + 1}$. The general expressions for IE^2, ITE^2, and IT^2E^2 simplify to

$$IE^2 = \zeta + \frac{1}{4\zeta} \tag{44}$$

$$ITE^2 = \zeta^2 + \frac{1}{8\zeta^2} \tag{45}$$

$$IT^2E^2 = \frac{1}{8\zeta^3} + \frac{1}{8\zeta} + 2\zeta^3 - \frac{1}{2}\zeta \tag{46}$$

These measures have been evaluated and graphed in Fig. 16.

The graphs of Ref. 37, which were obtained by mechanization of analog computer responses, are also shown in Fig. 16 for comparative purposes. The discrepancies in these curves are likely to be attributable to

1. A scaling error of a factor of 10 for ITE^2
2. A scaling error of approximately 16 for IT^2E^2
3. Amplifier drift for $\zeta > 0.9$ for all three measures

The conclusions of Ref. 37 regarding the value of these measures as criteria for normalized second-order systems need not be changed by the correction of these errors. IE^2 is minimized by $\zeta = 0.5$, which is not obviously undesirable, although rather higher damping ratios ($\zeta = 0.7$) are usually preferred for step responses.

The strongest objection to IE^2 as a performance measure for second-order systems is its lack of selectivity. Varying ζ from 0.2 to 1.3 raises IE^2 from its minimum value of 1 to only 1.5. By contrast, ITE^2, which has a minimum value of 0.9 at $\zeta = 0.67$, increases 50 percent (to 1.35) at $\zeta = 0.31$ and $\zeta = 0.9$. For the normalized third-order system investigated in Ref. 37 with the transfer function

$$\frac{C(s)}{R(s)} = \frac{1}{s^3 + bs^2 + cs + 1} \tag{47}$$

TABLE VI

STANDARD FORMS FOR UNIT-NUMERATOR SYSTEMS
(NUMERATOR IS Ω_0^n, WHERE n IS SYSTEM ORDER)

CLOSED-LOOP GENERAL POLYNOMIAL	FACTORED CLOSED-LOOP
	THE MINIMUM $\int_0^\infty e^2 dt$
$s + \Omega_0$ $s^2 + 1.000\Omega_0 s + \Omega_0^2$ $s^3 + 1.000\Omega_0 s^2 + 2.000\Omega_0^2 s + \Omega_0^3$ $s^4 + 1.000\Omega_0 s^3 + 2.000\Omega_0^2 s^2 + 2.000\Omega_0^3 s + \Omega_0^4$	$s + \Omega_0$ $s^2 + 2(.5)\Omega_0 s + \Omega_0^2$ $(s + .570\Omega_0)[s^2 + 2(.164)(1.325\Omega_0)s + (1.325\Omega_0$ $[s^2 + 2(.615)(.642\Omega_0)s + (.642\Omega_0)^2][s^2 + 2(.068)(1.556\Omega_0)$
	THE MINIMUM $\int_0^\infty te^2 dt$
$s + \Omega_0$ $s^2 + 1.189\Omega_0 s + \Omega_0^2$ $s^3 + 1.290\Omega_0 s^2 + 2.000\Omega_0^2 s + \Omega_0^3$ $s^4 + 1.278\Omega_0 s^3 + 3.012\Omega_0^2 s^2 + 2.209\Omega_0^3 s + \Omega_0^4$	$s + \Omega_0$ $s^2 + 2(.595)\Omega_0 s + \Omega_0^2$ $(s + .620\Omega_0)[s^2 + 2(.249)(1.270\Omega_0)s + (1.270\Omega_0)^2$ $[s^2 + 2(.685)(.669\Omega_0)s + (.669\Omega_0)^2][s^2 + 2(.121)(1.494\Omega_0)$
	THE MINIMUM $\int_0^\infty t^2 e^2 dt$
$s + \Omega_0$ $s^2 + 1.336\Omega_0 s + \Omega_0^2$ $s^3 + 1.472\Omega_0 s^2 + 2.042\Omega_0^2 s + \Omega_0^3$ $s^4 + 1.559\Omega_0 s^3 + 3.072\Omega_0^2 s^2 + 2.372\Omega_0^3 s + \Omega_0^4$	$s + \Omega_0$ $s^2 + 2(.668)\Omega_0 s + \Omega_0^2$ $(s + .665\Omega_0)[s^2 + 2(.322)(1.227\Omega_0)s + (1.227\Omega_0$ $[s^2 + 2(.74)(.696\Omega_0)s + (.696\Omega_0)^2][s^2 + 2(.177)(1.457\Omega_0)$
	THE MINIMUM $\int_0^\infty t^3 e^2 dt$
$s + \Omega_0$ $s^2 + 1.448\Omega_0 s + \Omega_0^2$ $s^3 + 1.662\Omega_0 s^2 + 2.094\Omega_0^2 s + \Omega_0^3$ $s^4 + 1.185\Omega_0 s^3 + 3.177\Omega_0^2 s^2 + 2.514\Omega_0^3 s + \Omega_0^4$	$s + \Omega_0$ $s^2 + 2(.724)\Omega_0 s + \Omega_0^2$ $(s + .704\Omega_0)[s^2 + 2(.402)(1.192\Omega_0)s + (1.192$ $[s^2 + 2(.785)(.722\Omega_0)s + (.722\Omega_0)^2][s^2 + 2(.235)(1.385\Omega_0$
	THE MINIMUM $\int_0^\infty t\lvert e\rvert dt$
$s + \Omega_0$ $s^2 + 1.52\Omega_0 s + \Omega_0^2$ $s^3 + 1.75\Omega_0 s^2 + 2.15\Omega_0^2 s + \Omega_0^3$ $s^4 + 2.1\Omega_0 s^3 + 3.4\Omega_0^2 s^2 + 2.7\Omega_0^3 s + \Omega_0^4$	$s + \Omega_0$ $s^2 + 2(.76)\Omega_0 s + \Omega_0^2$ $(s + .708\Omega_0)[s^2 + 2(.439)(1.188\Omega_0)s + (1.188\Omega_0$ $[s^2 + 2(.318)(1.333\Omega_0)s + (1.333\Omega_0)^2][s^2 + 2(.824)(.750\Omega_0$
	MINIMUM SETTLING TIME CRITERION
$s + \Omega_0$ $s^2 + 1.4\Omega_0 s + \Omega_0^2$ $s^3 + 1.55\Omega_0 s^2 + 2.10\Omega_0^2 s + \Omega_0^3$ $s^4 + 1.60\Omega_0 s^3 + 3.15\Omega_0^2 s^2 + 2.45\Omega_0^3 s + \Omega_0^4$	$s + \Omega_0$ $s^2 + 2(.7)\Omega_0 s + \Omega_0^2$ $(s + .661\Omega_0)[s^2 + 2(.361)(1.230\Omega_0)s + (1.230\Omega_0)^2$ $[s^2 + 2(.755)(.691\Omega_0)s + (.691\Omega_0)^2][s^2 + 2(.192)(1.447\Omega_0)$
	THE BUTTERWORTH
$s + \Omega_0$ $s^2 + 1.4\Omega_0 s + \Omega_0^2$ $s^3 + 2.0\Omega_0 s^2 + 2.0\Omega_0^2 s + \Omega_0^3$ $s^4 + 2.6\Omega_0 s^3 + 3.4\Omega_0^2 s^2 + 2.6\Omega_0^3 s + \Omega_0^4$	$s + \Omega_0$ $s^2 + 2(.7)\Omega_0 s + \Omega_0^2$ $(s + \Omega_0)\ [s^2 + 2(.5)\Omega_0 s + \Omega_0^2]$ $[s^2 + 2(.92)\Omega_0 s + \Omega_0^2][s^2 + 2(.38)\Omega_0 s + \Omega_0^2]$
	THE BINOMIAL
$s + \Omega_0$ $s^2 + 2\Omega_0 s + \Omega_0^2$ $s^3 + 3\Omega_0 s^2 + 3\Omega_0^2 s + \Omega_0^3$ $s^4 + 4\Omega_0 s^3 + 6\Omega_0^2 s^2 + 4\Omega_0^3 s + \Omega_0^4$	$(s + \Omega_0)$ $(s + \Omega_0)^2$ $(s + \Omega_0)^3$ $(s + \Omega_0)^4$

1

TABLE VI

STANDARD FORMS FOR UNIT-NUMERATOR SYSTEMS
(NUMERATOR IS Ω_0^n, WHERE n IS SYSTEM ORDER)

FACTORED CLOSED-LOOP

THE MINIMUM $\int_0^\infty e^2 dt$

$$s + \Omega_0$$
$$s^2 + 2(.5)\Omega_0 s + \Omega_0^2$$
$$(s + [illegible]\Omega_0)\left[s^2 + 2(.164)(1.32[illegible]\Omega_0)s + (1.32[illegible]\Omega_0)^2\right]$$
$$\left[s^2 + 2(.61[illegible])(.642\Omega_0)s + (.642\Omega_0)^2\right]\left[s^2 + 2(.068)(1.556\Omega_0)s + (1 \ldots\right.$$

THE MINIMUM $\int_0^\infty te^2 dt$

$$s + \Omega_0$$
$$s^2 + 2(.595)\Omega_0 s + \Omega_0^2$$
$$(s + .620\Omega_0)\left[s^2 + 2(.245)(1.270\Omega_0)s + (1.270\Omega_0)^2\right]$$
$$\left[s^2 + 2(.685)(.660\Omega_0)s + (.660\Omega_0)^2\right]\left[s^2 + 2(.121)(1.494\Omega_0)s + (1 \ldots\right.$$

THE MINIMUM $\int_0^\infty t^2e^2 dt$

$$s + \Omega_0$$
$$s^2 + 2(.668)\Omega_0 s + \Omega_0^2$$
$$(s + .660\Omega_0)\left[s^2 + 2(.329)(1.227\Omega_0)s + (1.227\Omega_0)^2\right]$$
$$\left[s^2 + 2(.74)(.696\Omega_0)s + (.696\Omega_0)^2\right]\left[s^2 + 2(.177)(1.457\Omega_0)s + (1 \ldots\right.$$

THE MINIMUM $\int_0^\infty t^3e^2 dt$

$$s + \Omega_0$$
$$s^2 + 2(.724)\Omega_0 s + \Omega_0^2$$
$$(s + .704\Omega_0)\left[s^2 + 2(.402)(1.192\Omega_0)s + (1.192\Omega_0)^2\right]$$
$$\left[s^2 + 2(.785)(.722\Omega_0)s + (.722\Omega_0)^2\right]\left[s^2 + 2(.255)(1.385\Omega_0)s + (1 \ldots\right.$$

THE MINIMUM $\int_0^\infty t|e|dt$

$$s + \Omega_0$$
$$s^2 + 2(.76)\Omega_0 s + \Omega_0^2$$
$$(s + .708\Omega_0)\left[s^2 + 2(.439)(1.188\Omega_0)s + (1.188\Omega_0)^2\right]$$
$$\left[s^2 + 2(.318)(1.333\Omega_0)s + (1.333\Omega_0)^2\right]\left[s^2 + 2(.824)(.750\Omega_0)s + (\ldots\right.$$

MINIMUM SETTLING TIME CRITERION

$$s + \Omega_0$$
$$s^2 + 2(.7)\Omega_0 s + \Omega_0^2$$
$$(s + .661\Omega_0)\left[s^2 + 2(.361)(1.25\Omega_0)s + (1.25\Omega_0)^2\right]$$
$$\left[s^2 + 2(.755)(.691\Omega_0)s + (.691\Omega_0)^2\right]\left[s^2 + 2(.192)(1.447\Omega_0)s + (1 \ldots\right.$$

THE BUTTERWORTH

$$s + \Omega_0$$
$$s^2 + 2(.7)\Omega_0 s + \Omega_0^2$$
$$(s + \Omega_0)\left[s^2 + 2(.5)\Omega_0 s + \Omega_0^2\right]$$
$$\left[s^2 + 2(.92)\Omega_0 s + \Omega_0^2\right]\left[s^2 + 2(.38)\Omega_0 s + \Omega_0^2\right]$$

THE BINOMIAL

$$(s + \Omega_0)$$
$$(s + \Omega_0)^2$$
$$(s + \Omega_0)^3$$
$$(s + \Omega_0)^4$$

2

FACTORED CLOSED-LOOP	OPEN-LOOP
THE MINIMUM $\int_0^\infty e^2dt$ $s + \Omega_0$ $s^2 + 2(.[illegible])\Omega_0 s + \Omega_0^2$ $(s + .[illegible]\Omega_0)[s^2 + 2(.[illegible])(1.[illegible]\Omega_0)s + (1.[illegible]\Omega_0)^2]$ $[s^2 + 2(.[illegible])(.[illegible]\Omega_0)s + (.[illegible]\Omega_0)^2][s^2 + 2(.008)(1.[illegible]\Omega_0)s + (1.[illegible]\Omega_0)^2]$	s $s(s + \Omega_0)$ $s[s^2 + 2(.[illegible])(1.414\Omega_0)s + (1.414\Omega_0)^2]$ $s[s^3 + \Omega_0 s^2 + 3\Omega_0^2 s + 2\Omega_0^3]$
THE MINIMUM $\int_0^\infty te^2dt$ $s + \Omega_0$ $s^2 + 2(.[illegible])\Omega_0 s + \Omega_0^2$ $(s + .[illegible]\Omega_0)[s^2 + 2(.2[illegible])(1.27[illegible]\Omega_0)s + (1.27[illegible]\Omega_0)^2]$ $[s^2 + 2(.[illegible])(.[illegible]\Omega_0)s + (.[illegible]\Omega_0)^2][s^2 + 2(.121)(1.494\Omega_0)s + (1.4[illegible]\Omega_0)^2]$	s $s(s + 1.1[illegible]\Omega_0)$ $s[s^2 + 2(.443)(1.416\Omega_0)s + (1.416\Omega_0)^2]$ $s[s^3 + 1.278\Omega_0 s^2 + 3.012\Omega_0^2 s + 2.20[illegible]\Omega_0^3]$
THE MINIMUM $\int_0^\infty t^2e^2dt$ $s + \Omega_0$ $s^2 + 2(.608)\Omega_0 s + \Omega_0^2$ $(s + .60[illegible]\Omega_0)[s^2 + 2(.[illegible])(1.227\Omega_0)s + (1.227\Omega_0)^2]$ $[s^2 + 2(.[illegible])(.[illegible]\Omega_0)s + (.[illegible]\Omega_0)^2][s^2 + 2(.177)(1.457\Omega_0)s + (1.457\Omega_0)^2]$	s $s(s + 1.55[illegible]\Omega_0)$ $s[s^2 + 2(.515)(1.420\Omega_0)s + (1.420\Omega_0)^2]$ $s[s^3 + 1.559\Omega_0 s^2 + 3.072\Omega_0^2 s + 2.572\Omega_0^3]$
THE MINIMUM $\int_0^\infty t^{[illegible]}e^2dt$ $s + \Omega_0$ $s^2 + 2(.724)\Omega_0 s + \Omega_0^2$ $(s + .704\Omega_0)[s^2 + 2(.402)(1.192\Omega_0)s + (1.192\Omega_0)^2]$ $[s^2 + 2(.7[illegible])(.722\Omega_0)s + (.722\Omega_0)^2][s^2 + 2(.255)(1.385\Omega_0)s + (1.385\Omega_0)^2]$	s $s(s + 1.448\Omega_0)$ $s[s^2 + 2(.575)(1.445\Omega_0)s + (1.445\Omega_0)^2]$ $s[s^3 + 1.785\Omega_0 s^2 + 3.177\Omega_0^2 s + 2.[illegible]\Omega_0^3]$
THE MINIMUM $\int_0^\infty t\lvert e\rvert dt$ $s + \Omega_0$ $s^2 + 2(.76)\Omega_0 s + \Omega_0^2$ $(s + .708\Omega_0)[s^2 + 2(.45[illegible])(1.188\Omega_0)s + (1.188\Omega_0)^2]$ $[s^2 + 2(.518)(1.333\Omega_0)s + (1.333\Omega_0)^2][s^2 + 2(.824)(.750\Omega_0)s + (.750\Omega_0)^2]$	s $s(s + 1.52\Omega_0)$ $s[s^2 + 2(.685)(1.467\Omega_0)s + (1.467\Omega_0)^2]$ $s[s^3 + 2.1\Omega_0 s^2 + 3.4[illegible]\Omega_0^2 s + 2.7[illegible]\Omega_0^3]$
MINIMUM SETTLING TIME CRITERION $s + \Omega_0$ $s^2 + 2(.7[illegible])\Omega_0 s + \Omega_0^2$ $(s + .661\Omega_0)[s^2 + 2(.[illegible]61)(1.230\Omega_0)s + (1.230\Omega_0)^2]$ $[s^2 + 2(.7[illegible])(.691\Omega_0)s + (.691\Omega_0)^2][s^2 + 2(.192)(1.447\Omega_0)s + (1.447\Omega_0)^2]$	s $s(s + 1.4[illegible]\Omega_0)$ $s[s^2 + 2(.5[illegible])(1.450\Omega_0)s + (1.450\Omega_0)^2]$ $s[s^3 + 1.600\Omega_0 s^2 + 3.15[illegible]\Omega_0^2 s + 2.45[illegible]\Omega_0^3]$
THE BUTTERWORTH $s + \Omega_0$ $s^2 + 2(.7)\Omega_0 s + \Omega_0^2$ $(s + \Omega_0)[s^2 + 2(.5)\Omega_0 s + \Omega_0^2]$ $[s^2 + 2(.92)\Omega_0 s + \Omega_0^2][s^2 + 2(.38)\Omega_0 s + \Omega_0^2]$	s $s(s + 1.4\Omega_0)$ $s[s^2 + 2(.707)(1.414\Omega_0)s + (1.414\Omega_0)^2]$ $s(s + 1.49\Omega_0)[s^2 + 2(.42)(1.32\Omega_0) + (1.32\Omega_0)^2]$
THE BINOMIAL $(s + \Omega_0)$ $(s + \Omega_0)^2$ $(s + \Omega_0)^3$ $(s + \Omega_0)^4$	s $s(s + 2\Omega_0)$ $s[s^2 + 2(.866)(1.732\Omega_0)s + (1.732\Omega_0)^2]$ $s(s + 2\Omega_0)[s^2 + 2(.707)(1.414\Omega_0)s + (1.414\Omega_0)^2]$

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TABLE VII

FREQUENCY-DOMAIN CHARACTERISTICS OF STANDARD FO

DESIGNATION	CRITERION	SYSTEM ORDER	CLOSED-LOOP ω_b	CLOSED-LOOP M_p, db
Butterworth	---------	2 3 4	$1.00\ \Omega_0$ $1.05\ \Omega_0$ $1.00\ \Omega_0$	0 0 0
ITAE	Minimum $\int_0^\infty t\lvert e\rvert dt$	2 3 4	$1.00\ \Omega_0$ $1.05\ \Omega_0$ $0.95\ \Omega_0$	0 0 0
----	Minimum Settling Time	2 3 4	$1.00\ \Omega_0$ $1.14\ \Omega_0$ $0.80\ \Omega_0$	0 0 0
IE^2	Minimum $\int_0^\infty e^2 dt$	2 3 4	$1.28\ \Omega_0$ $1.52\ \Omega_0$ $1.67\ \Omega_0$	1.3 1.5 2
ITE^2	Minimum $\int_0^\infty te^2 dt$	2 3 4	$1.16\ \Omega_0$ $1.40\ \Omega_0$ $1.58\ \Omega_0$	0.5 0 0
IT^2E^2	Minimum $\int_0^\infty t^2e^2 dt$	2 3 4	$1.00\ \Omega_0$ $1.15\ \Omega_0$ $1.35\ \Omega_0$	0 0 0.25
IT^3E^2	Minimum $\int_0^\infty t^3e^2 dt$	2 3 4	$1.00\ \Omega_0$ $1.05\ \Omega_0$ $1.00\ \Omega_0$	0 0 0

1

TABLE VII

FREQUENCY-DOMAIN CHARACTERISTICS OF STANDARD FORMS

...ION	SYSTEM ORDER	CLOSED-LOOP		OPEN-LOOP		
		ω_b	M_p, db	ω_c	φ_m, deg	Gain Margin, db
	2	1.00 Ω_0	0	0.67 Ω_0	64	----
---	3	1.05 Ω_0	0	0.49 Ω_0	60	13
	4	1.00 Ω_0	0	0.38 Ω_0	60	8
∞	2	1.00 Ω_0	0	0.64 Ω_0	67	----
$t\|e\|dt$	3	1.05 Ω_0	0	0.48 Ω_0	66	11.5
	4	0.95 Ω_0	0	0.37 Ω_0	60	8.5
	2	1.00 Ω_0	0	0.67 Ω_0	64	----
tling Time	3	1.14 Ω_0	0	0.50 Ω_0	67	10.2
	4	0.80 Ω_0	0	0.37 Ω_0	60	8.6
∞	2	1.28 Ω_0	1.3	0.77 Ω_0	52	----
e^2dt	3	1.52 Ω_0	1.5	0.56 Ω_0	71	6
	4	1.67 Ω_0	2	0.45 Ω_0	53	6.5
∞	2	1.16 Ω_0	0.5	0.75 Ω_0	56	----
te^2dt	3	1.40 Ω_0	0	0.55 Ω_0	68	8
	4	1.58 Ω_0	0	0.40 Ω_0	60	6.5
∞	2	1.00 Ω_0	0	0.66 Ω_0	62	----
t^2e^2dt	3	1.15 Ω_0	0	0.49 Ω_0	68	9.5
	4	1.35 Ω_0	0.25	0.41 Ω_0	58	7.8
∞	2	1.00 Ω_0	0	0.64 Ω_0	66	----
t^3e^2dt	3	1.05 Ω_0	0	0.46 Ω_0	66	11
	4	1.00 Ω_0	0	0.38 Ω_0	64	8.3

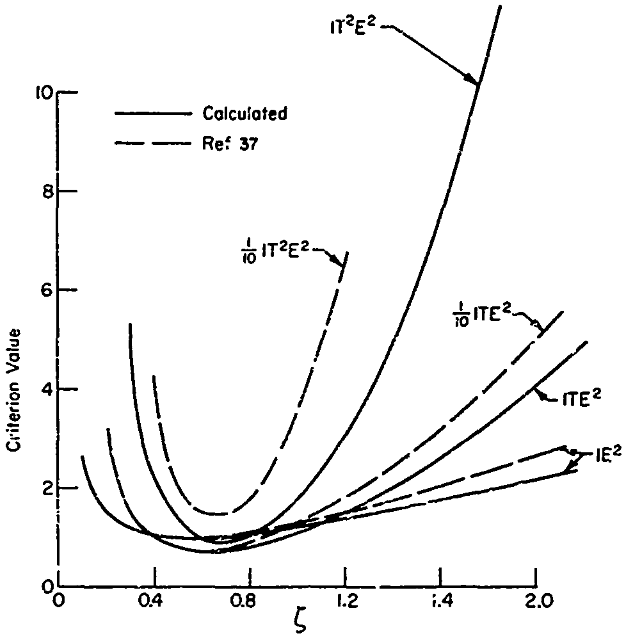

Figure 16. Comparison of Analytical and Experimentally Determined IE^2, ITE^2, and IT^2E^2 Performance Measures

IE^2 becomes even less selective. As shown in Fig. 11 of Ref. 37, for b ranging from 1.25 to 5.0, c can be varied from 1.5 to 3.0 with less than 10 percent resulting variation in IE^2. The indicial responses of Fig. 15 indicate that minimization of IE^2 selects oscillatory responses for higher-order systems.

An advantage of IE^2 is that its analytic expressions are simpler than those for ITE^2 and IT^2E^2. The last measure is highly selective, but it is doubtful whether the added complication of the general forms is worth the extra computational effort required, compared to ITE^2.

Having stated the advantages and disadvantages of IE^2, ITE^2, and IT^2E^2, some alternative criteria can now be assessed to arrive at a unique selection, if such a selection can be made. The next section of this chapter is concerned with IAE, ITAE, and IT^2AE, but it would be misleading to close the discussion of IE^2, ITE^2, etc., without a brief digression relating to the suitability of these criteria for statistical inputs.

A strong argument in favor of IE^2 is that, for statistical inputs, the corresponding measure $\overline{e^2(t)} = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{+T} [e(t)]^2 dt$ has been thoroughly investigated, and forms the foundation for an extensive literature, including many of the most valuable and widely used contributions to optimization theory.

Statistical inputs fall outside the scope of the present report. However, it would be unrealistic to ignore the fact that flight control systems are subjected to both statistical and deterministic inputs, and it would be unfortunate if the already considerable gap between random and deterministic methods of analysis was widened by demands for different criteria for each type of input. It is suggested that the possibility of developing a compatible criterion might well form the subject of a future investigation. However, for the remainder of this report, statistical considerations will be disregarded, and assessments of indicial error measures will be made with respect only to their suitability for the task of measuring system performance following a step input.

DERIVATION OF IAE, ITAE, AND IT^2AE

The IAE, ITAE, and IT^2AE measures have been investigated in Ref. 37, 38, and 39. Reference 37 includes an extensive study of the properties of ITAE obtained by mechanization of analog computer responses for unit-numerator zero-position-error systems of first- through eighth-order, first-order numerator zero-velocity-

... systems of second- through sixth-order, and zero-acceleration-error systems of third- through sixth-order. Indicial responses and standard forms for each of these systems are presented. Some data are also given on IAE and IT²AE, but ITAE is emphasized because it is more selective than IAE, and is easier to apply than IT²AE. (ITAE can be easily mechanized and scaled on an analog computer, e.g., using the simple circuit described in Ref. 37.)

The present report continues the investigation of these measures. Exact analytic expressions for IAE, ITAE, and IT²AE of second-order systems are presented, and the results are employed to check those of Ref. 37. As will be shown, some errors are detected. An analytic method for obtaining IAE, ITAE, and IT²AE for higher-order systems is also outlined. In addition, Chapter IV describes a procedure for the approximate calculation of ITAE for near-optimum systems.

The procedure for calculating IAE, ITAE, and IT²AE for a second-order system is detailed in Appendix A, and briefly summarized below.

The starting point is the calculation of the Laplace transform of the absolute error time history, $E_A(s)$. Once this has been obtained, IAE, ITAE, and IT²AE are readily obtained by means of the final value theorem, as indicated below.

$$\text{IAE} = \lim_{t_1 \to \infty} \int_0^{t_1} |e(t)|\,dt = \lim_{s \to 0} s\,\frac{E_A(s)}{s} \tag{48}$$

$$\text{ITAE} = \lim_{t_1 \to \infty} \int_0^{t_1} t|e(t)|\,dt = \lim_{s \to 0} s\,\frac{1}{s}\,\frac{-dE_A(s)}{ds} \tag{49}$$

$$\text{IT}^2\text{AE} = \lim_{t_1 \to \infty} \int_0^{t_1} t^2|e(t)|\,dt = \lim_{s \to 0} s\,\frac{1}{s}\,\frac{d^2E_A(s)}{ds^2} \tag{50}$$

The method for obtaining $E_A(s)$ analytically is an extension of the technique used for obtaining the Laplace transform of a rectified sine wave. The error time history is a damped sine wave with a phase lag

$$e(t) = \frac{\omega_n e^{-\zeta\omega_n t}}{\beta} \sin\,(\beta t + \psi) \tag{51}$$

$$e(t) = \frac{1}{\sqrt{1 - \zeta^2}}\, e^{-\zeta\omega_n t} \sin\left(\omega_n \sqrt{1 - \zeta^2}\, t + \psi\right) \tag{52}$$

where
$$\psi = \sin^{-1}\sqrt{1-\zeta^2}$$
$$\beta = \omega_n\sqrt{1-\zeta^2}$$

From the transform tables of Ref. 16,

$$\mathcal{L}|\sin\omega t| = \coth\frac{\pi}{2\omega}\mathcal{L}\sin\omega t = \left(\coth\frac{\pi s}{2\omega}\right)\left(\frac{\omega^2}{s^2+\omega^2}\right) \tag{53}$$

As shown in Appendix A, the damping factor $e^{-\zeta\omega_n t}$ in Eq 52 can be accounted for by replacing s with $s + \zeta\omega_n$, to obtain $\mathcal{L}|e^{-\zeta\omega_n t}\sin\omega t|$. The phase lag introduces some complications, but the final result is reasonably compact.

$$E_A(s) = \frac{\omega_n e^{\psi(s+\zeta\omega_n)/\omega}}{(s+\zeta\omega_n)^2+\omega^2}\left[\coth\frac{\pi(s+\zeta\omega_n)}{2\omega} - 1\right] + \frac{s+2\zeta\omega_n}{(s+\zeta\omega_n)^2+\omega^2} \tag{54}$$

where

$$\omega = \omega_n\sqrt{1-\zeta^2}$$

Note that the last term in Eq 54 is simply E(s).

The procedure for obtaining an analytical expression for $|e(t)|$, or $E_A(s)$, for general higher-order systems is more complicated. For these systems, the the zero crossings of the error time history are not equally spaced, and Eq 53 does not apply. The procedure used in Appendix B for a third-order system having a complex pair of roots was to represent the error response by a Fourier-like series, the coefficients of which are time-dependent. The final result is quite complicated. It provides a basis for further theoretical studies, and for checking experimentally obtained IAE, etc., but seems too involved for routine optimization calculations.

For a normalized unit-numerator second-order system described by Eq 29, the IAE and ITAE are given by

$$\mathrm{IAE} = \frac{e^{\frac{\zeta}{\sqrt{1-\zeta^2}}\cos^{-1}\zeta}}{\omega_n}\left[\coth\left(\frac{\pi\zeta}{2\sqrt{1-\zeta^2}}\right) - 1\right] + \frac{2\zeta}{\omega_n} \tag{55}$$

and

and

$$\mathrm{ITAE} = \left(\coth\left[\frac{\pi\zeta}{2\sqrt{1-\zeta^2}}\right] - 1\right)\left(4\zeta\sqrt{1-\zeta^2} + \pi \coth \frac{\pi\zeta}{2\sqrt{1-\zeta^2}} + 2 \sin^{-1} \zeta\right)$$

$$\times \left(\frac{e^{\frac{\zeta}{\sqrt{1-\zeta^2}} \cos^{-1}\zeta}}{2\omega_n^2\sqrt{1-\zeta^2}}\right) + \frac{4\zeta^2 - 1}{\omega_n^2} \tag{56}$$

These expressions, and Eq A-40 of Appendix A (which involves IT^2AE) are presented in Fig. 17 and 18, and are compared with the values obtained by Graham and Lathrop in Ref. 37. It is evident that a scaling error of two exists in the Graham and Lathrop graph for IT^2AE. Agreement on IAE is excellent; some minor discrepancies exist in ITAE. Minimization occurs at $\zeta = 0.76$ instead of $\zeta = 0.7$, but this error is insignificant.

The conclusions of Graham and Lathrop regarding the relative merits of these criteria are not altered by the present study. As would be anticipated, IAE, ITAE, and IT^2AE exhibit similar characteristics to IE^2, ITE^2, and IT^2E^2, respectively. IAE is unselective, and IT^2AE is too complicated; therefore, ITAE is preferred. ITAE has been explored more thoroughly than any other indicial error measure, and the considerable background of knowledge established by Ref. 37 and 38 permits this measure to be used with confidence. For each case investigated in Ref. 37, minimum ITAE yielded "good" responses, and the criterion remained highly selective for systems of third-, fourth-, and fifth-order.

Compared with ITE^2, ITAE has the advantages of giving less oscillatory indicial standard forms and of being more thoroughly explored, but it is the more difficult of the two measures to express analytically. The choice between these criteria must depend on the circumstances of the individual cases in which they are to be applied. Possibly ITE^2 would be more convenient for an analytical investigation, and ITAE best for optimization using analog computers, because of the ease with which it can be mechanized and scaled.

Closed-loop Bode diagrams of the ITAE standard forms listed in Table VI are given in Fig. 14. Associated indicial responses are given in Fig. 15. Table VIII

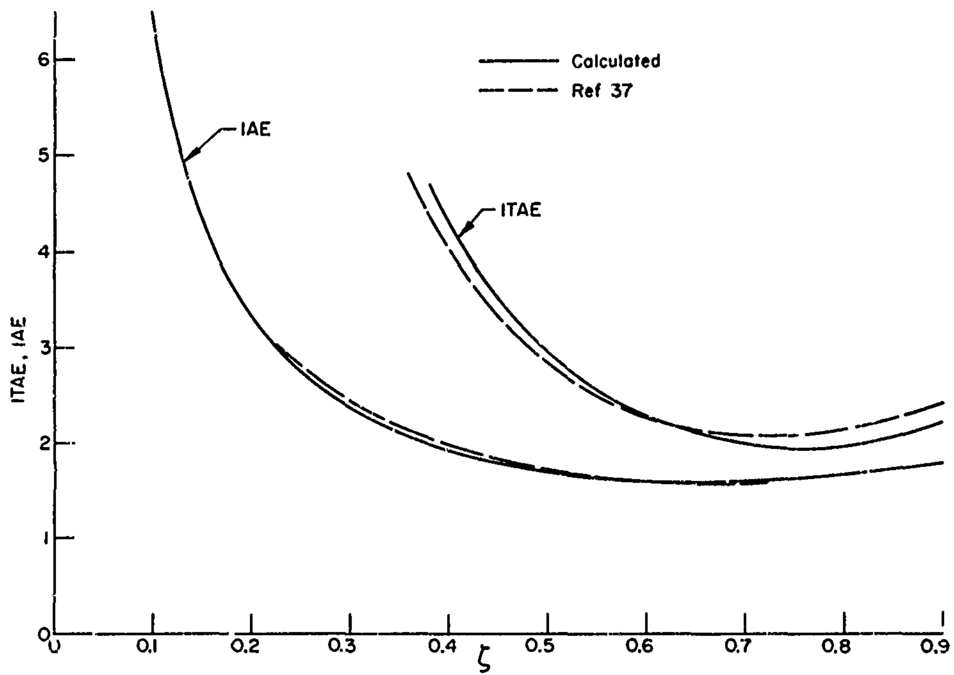

Figure 17. Comparison of Analytical and Experimentally Determined IAE and ITAE for Unit-Numerator Second-Order System

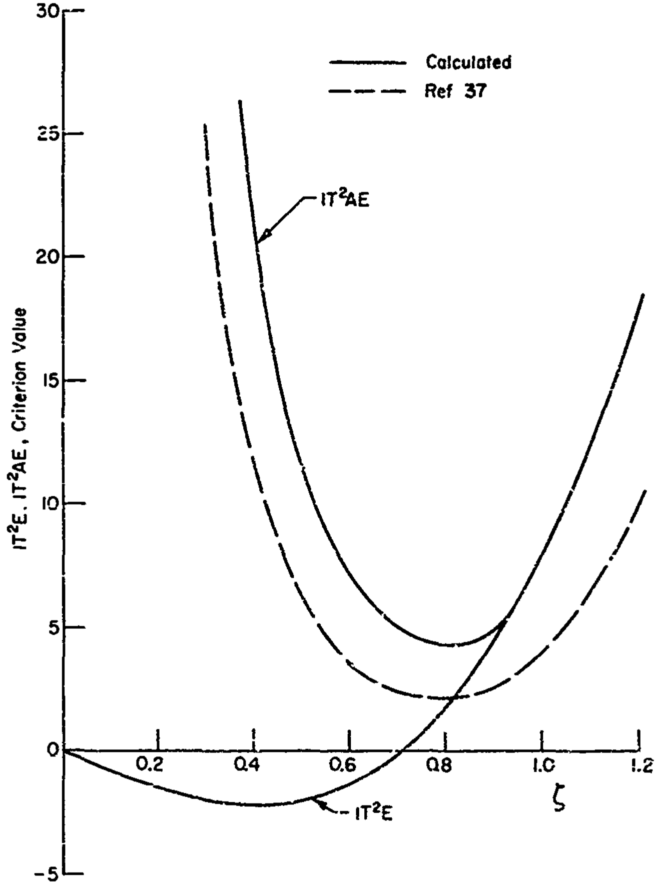

Figure 18. Comparison of Analytical and Experimentally Determined IT^2AE for Unit-Numerator Second-Order System

TABLE VIII

INDICIAL RESPONSE CHARACTERISTICS OF OPTIMAL SYSTEMS

	Delay Time	Rise Time	Time to Peak	Peak Overshoot	Equivalent Time Constant	Settling Time
<u>Second-Order</u>						
Minimum Settling Time	1.43	2.24	4.40	0.046	1.74	2.96
$\int_0^\infty t\|e(t)\|dt$	1.51	2.54	5.03	0.023	1.86	3.47
$\int_0^\infty e(t)^2 dt$	1.32	1.65	3.62	0.165	1.56	5.34
$\int_0^\infty t\, e(t)^2 dt$	1.4	1.84	3.93	0.096	1.67	5.26
$\int_0^\infty t^2 e(t)^2 dt$	1.44	2.00	4.25	0.062	1.72	5.03
$\int_0^\infty t^3 e(t)^2 dt$	1.5	2.21	4.6	0.041	1.81	3.05
<u>Third-Order</u>						
Minimum Settling Time	2.02	2.08	4.32	0.031	2.32	3.55
$\int_0^\infty t\|e(t)\|dt$	2.11	2.3	4.67	0.029	2.44	3.58
$\int_0^\infty e(t)^2 dt$	1.83	1.76	3.62	0.071	2.10	6.71
$\int_0^\infty t\, e(t)^2 dt$	1.94	1.93	3.90	0.061	2.21	6.98
$\int_0^\infty t^2 e(t)^2 dt$	2.0	2.32	4.16	0.043	2.30	6.82
$\int_0^\infty t^3 e(t)^2 dt$	2.03	2.20	4.56	0.038	2.37	3.41
<u>Fourth-Order</u>						
Minimum Settling Time	2.56	2.17	4.78	0.05	2.88	3.8
$\int_0^\infty t\|e(t)\|dt$	2.62	2.47	5.44	0.027	3.08	4.23
$\int_0^\infty e(t)^2 dt$	2.40	1.83	7.75	0.141	2.68	10.32
$\int_0^\infty t\, e(t)^2 dt$	2.49	1.94	8.02	0.080	2.76	8.91
$\int_0^\infty t^2 e(t)^2 dt$	2.58	2.16	4.74	0.056	2.84	5.0
$\int_0^\infty t^3 e(t)^2 dt$	2.62	2.23	4.9	0.040	2.93	3.93

summarizes the indicial response characteristics of all the standard forms of Table VI. Standard form root locations are shown in Fig. 19, 20, and 21.

OTHER INDICIAL ERROR MEASURES

The measures discussed in the previous sections of this chapter have all been integrated functions of error. It is possible to generalize these measures by including functions of the time derivatives of error. Thus, several recent references (e.g., Ref. 79 and 80) have proposed optimization procedures based on the following very general measure:

$$\int_0^\infty F[e(t), t, \gamma_i]p(t)dt$$

Here $F[e(t), t, \gamma_i]$ is a general function of error, time, and system parameters γ_i, and p_i is the probability that the output will be used. The generality of this measure is both its advantage and its drawback. It cannot be put into concrete, usable form without making a number of arbitrary choices to arrive at F.

Consideration of the remaining measures listed in Table IV illustrates this point. The criterion

$$\int_0^\infty e^2(t)dt + \alpha\left(\int_0^\infty e^2(t)dt\right)^2 \int_0^\infty \left(\frac{de(t)}{dt}\right)^2 dt$$

yields, for a normalized second-order unit-numerator system, $\zeta = 0.866$ for $\alpha \to \infty$ and 0.5 for $\alpha \to 0$. (As shown in Ref. 73, the ζ selected by the criterion can fall outside the range 0.5 to 0.866 for a negative α.) No guarantee is available that an α suitable for a second-order system will yield acceptable responses when the same criterion is applied to a system of higher order. However, the criterion cannot be rejected on this basis alone without further investigation, particularly as its random analog should be easier to use with statistical inputs than the random analog of time-weighted criteria.

Arbitrary constants also appear in the quadratic form associated response measures listed at the bottom of Table IV, and in the very general measure

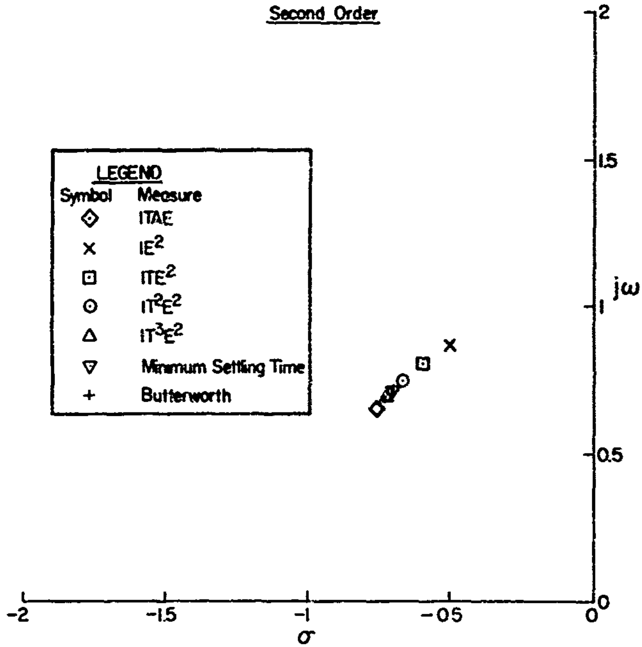

Figure 19. Pole Locations of Unit-Numerator Second-Order Standard Forms

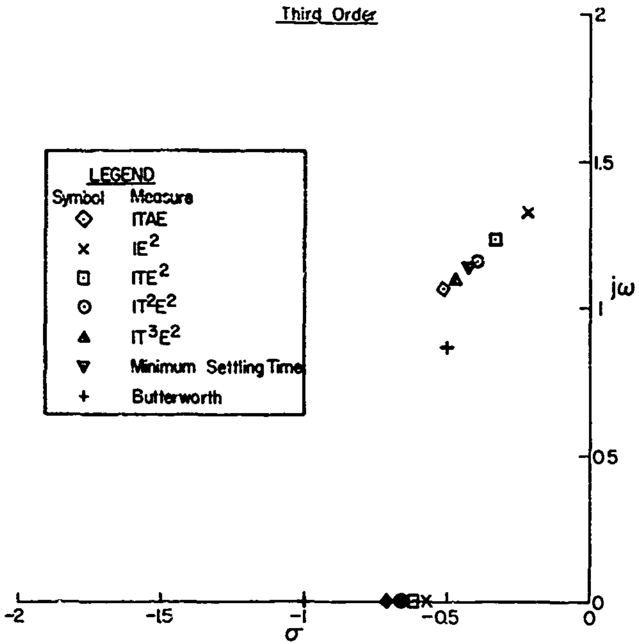

Figure 20. Pole Locations of Unit-Numerator Third-Order Forms

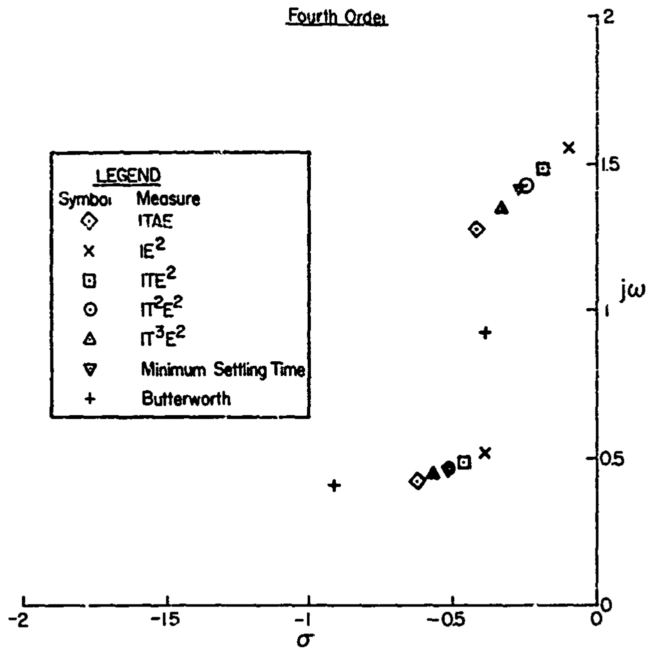

Figure 21. Pole Locations of Unit-Numerator Fourth-Order Forms

$$\int_0^\infty \sum_{n=0}^{N} \left\{ a_n \left[\frac{d^n e(t)}{dt^n}\right]^2 \right\} dt$$

Here the choice of a_1, a_2, ..., a_n is arbitrary, so that any desired weighting of the elements of the criterion may be used. Unfortunately, little guidance is available on the choice of a_1, a_2, ..., a_n and considerable effort may be required if it is necessary to evaluate the integrals. (However, this measure has usually been proposed in conjunction with dynamic programming procedures employing digital computers. For such applications, the latter disadvantage would be insignificant.)

New measures are constantly being added to the already vast literature on this subject.* It is perhaps worthwhile adding a plea for caution in accepting these measures as valid criteria. It is all too easy to present a "new" performance criterion which is valid, selective, and even readily applicable for some particular system. The task of formulating a criterion which retains these merits over the range of systems encountered in flight control design is much more difficult. Of the indicial error measures discussed in this chapter, ITAE stands out because of the thoroughness with which it has been investigated. The large number of examples given in the studies of Graham and Lathrop lend credence to their assertion that the ITAE criterion has exceptional merit.

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*This point is well-illustrated by Ref. 42, which was received while the present report was being typed. In addition to IAE, IE^2, ITAE, ITE, ITE^2, and IT^2AE, Ref. 42 investigates the following measures for normalized unit-numerator second- and third-order systems:

$$\int_0^\infty \sqrt{|e(t)|}\, dt, \int_0^\infty \sqrt{t|e(t)|}\, dt, \int_0^\infty t\sqrt{|e(t)|}\, dt,$$

$$\int_0^\infty t^2\sqrt{|e(t)|}\, dt. \text{ and } \int_0^\infty \frac{[e(t)]^2}{1 + [e(t)]^2}\, dt$$

Both digital and analog computers were employed to generate the measures. The results are generally in agreement with those of Ref. 70 and this report, except for errors (in Ref. 42) of approximately 10 percent on some standard form coefficients.

OTHER STANDARD FORMS

The standard forms discussed earlier in this chapter have all been optimal in the sense that they result in minimum values of particular indicial error criteria. Other criteria could be used for optimization, and Ref. 32 presents the standard forms associated with minimum settling time for systems of second- through eighth-order. The second-, third-, and fourth-order forms and associated transient responses are reproduced in Table VI and Fig. 15. Figure 14 illustrates the Bode (ω) diagrams for these systems. Table VI also lists binomial and Butterworth standard forms. The binomial forms are not optimal in the same sense as the standard forms considered previously, being generated by the following expression:

$$\frac{C(s)}{R(s)} = \frac{1}{(s + \Omega_0)^n} \tag{57}$$

where n is the system order.

The Butterworth forms are characterized by the fact that their poles are equally spaced around a semicircle of radius Ω_0 in the complex plane. The frequency-domain representation of the resulting transfer function is illustrated in Fig. 14 for second-, third-, and fourth-order systems, the indicial responses are given in Fig. 15. Butterworth forms are maximally flat, i.e., at $j\omega = 0$, the first n derivatives of the amplitude-frequency diagrams are zero, but their transient responses have increasingly large overshoots as the order of the system is increased. Reference 37 gives transient responses of the Butterworth filters for first- through eighth-order, which are reproduced in Fig. 28.

Neither the Butterworth nor the binomial standard forms are of direct use in flight control system optimization. However, they do provide a convenient starting point for devising other standard forms. The ITAE standard forms of Graham and Lathrop were, in fact, generated by systematically modifying Butterworth forms of the appropriate order.

CHAPTER IV

EFFECT OF A PURE TIME LAG UPON INDICIAL ERROR MEASURES

This chapter presents general exact formulae expressing the effect of system time lags on indicial error measures. These formulae may be used to obtain approximate relations expressing performance measures for some high-order systems in terms of the performance of "equivalent" lower-order systems possessing time lags.

As is usual, the analysis will be restricted to systems having zero steady-state error in response to a step-input, to obviate infinite values of indicial error measures. Three classes of indicial error measures will be considered, designated as in Eq 16, 17, 18 of Chapter III, and repeated below for convenience:

$$IU = \int_0^\infty U\,dt \qquad (16)$$

$$ITU = \int_0^\infty tU\,dt \qquad (17)$$

$$IT^2U = \int_0^\infty t^2U\,dt \qquad (18)$$

It will be shown that when a pure time delay, τ, is introduced, the measures can be expressed in terms of the measures appropriate to the undelayed system by means of the following formulae:

$$IU = U_0\tau + \left[IU\right]_{\tau=0} \qquad (58)$$

$$ITU = U_0\frac{\tau^2}{2} + \tau\left[IU\right]_{\tau=0} + \left[ITU\right]_{\tau=0} \qquad (59)$$

$$IT^2U = U_0\frac{\tau^3}{3} + \tau^2\left[IU\right]_{\tau=0} + 2\tau\left[ITU\right]_{\tau=0} + \left[IT^2U\right]_{\tau=0} \qquad (60)$$

where

$$U_0 = \left[U\right]_{\tau=0}$$

Proof:

IU, ITU, IT^2U, etc., can be calculated by application of the final value theorem. Thus,

$$IU = \lim_{s \to 0} s \cdot \frac{1}{s} \cdot U(s) \tag{61}$$

$$ITU = \lim_{s \to 0} s \cdot \frac{1}{s} \cdot -\frac{d}{ds} \cdot U(s) \tag{62}$$

$$IT^2U = \lim_{s \to 0} s \quad \frac{1}{s} \quad \frac{d^2}{ds^2} U(s) \tag{63}$$

For the system with lag,

$$U(s) = \frac{U_0}{s}(1 - e^{-\tau s}) + e^{-\tau s} U_{\tau=0}(s) \tag{64}$$

By the final value theorem, expanding $e^{-\tau s}$ as $1 - \tau s + \frac{(\tau s)^2}{2!} - \ldots$

$$IU = \int_0^\infty U dt = \lim_{s \to 0} s \cdot \frac{1}{s} \cdot U(s) - \lim_{s \to 0} U(s) \tag{65}$$

$$= \tau U_0 + \lim_{s \to 0} U_{\tau=0}(s) \tag{66}$$

$$= \tau U_0 + IU_{\tau=0} \tag{67}$$

which checks Eq 58.

ITU is evaluated similarly. From Eq 62 and 64,

$$\mathcal{L}\, tU(t) = -\frac{d}{ds} U(s) = -\frac{d}{ds}\left\{e^{-\tau s}\left[U_{\tau=0}(s) - \frac{U_0}{s}\right] + \frac{U_0}{s}\right\} \tag{68}$$

$$= e^{-\tau s}\left[-\frac{d}{ds} U_{\tau=0}(s)\right] - e^{-\tau s} \cdot \frac{U_0}{s^2}$$

$$+ \left[U_{\tau=0}(s) - \frac{U_0}{s}\right]\tau \cdot e^{-\tau s} + \frac{U_0}{s^2} \tag{69}$$

Applying the final value theorem

$$\int_0^{\infty} tU(t)dt = \lim_{s \to 0} e^{-\tau s}\left[-\frac{d}{ds}U_{\tau=0}(s)\right] + \lim_{s \to 0} \tau e^{-\tau s} U_{\tau=0}(s)$$

$$+ \lim_{s \to 0} \frac{U_0}{s^2}\left\{1 - e^{-\tau s}\right\} - \lim_{s \to 0} \frac{U_0}{s}\tau e^{-\tau s} \tag{70}$$

$$= \left[ITU\right]_{\tau=0} + \tau\left[IU\right]_{\tau=0} + \frac{\tau^2}{2}U_0 \tag{71}$$

which checks Eq 59.

To evaluate IT^2U, Eq 63 and 67 are combined to give

$$\mathcal{L}\, t^2U(t) = \frac{d^2}{ds^2}U(s) = \frac{d}{ds}\left\{e^{-\tau s}\frac{d}{ds}U_{\tau=0}(s) + e^{-\tau s}\frac{U_0}{s^2} - \tau e^{-\tau s}\left[U_{\tau=0}(s) - \frac{U_0}{s}\right] - \frac{U_0}{s^2}\right\} \tag{72}$$

$$= -\tau e^{-\tau s}\frac{d}{ds}U_{\tau=0}(s) + e^{-\tau s}\frac{d^2}{ds^2}U_{\tau=0}(s) - \frac{2U_0}{s^3}e^{-\tau s} - \tau\frac{U_0}{s^2}e^{-\tau s}$$

$$-\tau e^{-\tau s}\frac{d}{ds}U_{\tau=0}(s) + U_{\tau=0}(s)\cdot\tau^2 e^{-\tau s} - \tau e^{-\tau s}\frac{U_0}{s^2}$$

$$-\frac{U_0\tau^2}{s}e^{-\tau(s)} + \frac{2U_0}{s^3} \tag{73}$$

After some reduction, this yields

$$IT^2U = 2\tau\left[ITU\right]_{\tau=0} + \left[IT^2U\right]_{\tau=0} + \tau^2\left[IU\right]_{\tau=0} + U_0\frac{\tau^3}{3} \tag{74}$$

which checks Eq 60.

EFFECT OF TIME LAG ON ITAE AND ITE^2 FOR SECOND-ORDER ZERO-POSITION-ERROR SYSTEMS

The effects of a time lag, τ, on some performance measures for a second-order zero-position-error system are now considered. Apart from their intrinsic value, the results also provide approximations to performance measures of higher-order systems. The transfer function of a second-order system with lag applied directly to the input is

$$\frac{C(s)}{R(s)} = \frac{\omega_n^2 e^{-(\tau/\omega_n)s}}{s^2 + 2\zeta\omega_n s + \omega_n^2} \tag{75}$$

It is convenient to work in normalized (nondimensional) time so that

$$\frac{C(s)}{R(s)} = \frac{e^{-\tau s}}{s^2 + 2\zeta s + 1} \tag{76}$$

Denormalization of performance measures is easily accomplished by use of Eq 26, 27, and 28 of Chapter III.

For the system of Eq 75 with a step input

$$E(s) = e^{-\tau s} \frac{s + 2\zeta}{s^2 + 2\zeta s + 1} \tag{77}$$

Appendix A shows that for this system

$$\mathrm{IAE}_{\tau=0} = e^{\frac{\zeta}{\sqrt{1-\zeta^2}}\cos^{-1}\zeta} \left[\coth\left(\frac{\pi\zeta}{2\sqrt{1-\zeta^2}}\right) - 1\right] + 2\zeta \tag{78}$$

and

$$\mathrm{ITAE}_{\tau=0} = \left(\coth\frac{\pi\zeta}{2\sqrt{1-\zeta^2}} - 1\right)\left(4\zeta\sqrt{1-\zeta^2} + \pi\coth\frac{\pi\zeta}{2\sqrt{1-\zeta^2}} + 2\sin^{-1}\zeta\right)\frac{e^{\frac{\zeta}{\sqrt{1-\zeta^2}}\cos^{-1}\zeta}}{2\sqrt{1-\zeta^2}} + 4\zeta^2 - 1 \tag{79}$$

For $\zeta \geq 1$

$$[\mathrm{IAE}]_{\tau=0} = [\mathrm{IE}]_{\tau=0} = 2\zeta \tag{80}$$

$$[\mathrm{ITAE}]_{\tau=0} = [\mathrm{ITE}]_{\tau=0} = 4\zeta^2 - 1 \tag{81}$$

Substitution of these expressions into Eq 59 yields the results graphed in Fig. 22. The effect of time lag on ITE^2 has been calculated similarly. From Chapter III.

$$ITE^2 = \int_0^\infty t\left[e(t)\right]^2 dt = \zeta^2 + \frac{1}{8\zeta^2} \qquad (45)$$

The effect of various time delays on ITE^2 is shown in Fig. 23, which has been constructed by combining Eq 44, 45, and 59.

APPROXIMATION TO PERFORMANCE MEASURES OF HIGHER-ORDER OPTIMAL ZERO-POSITION-ERROR SYSTEMS

Figure 15 illustrates the indicial response time histories of optimum ITAE, ITE^2, and IT^{2}E^2 zero-position-error systems. These criteria may all be regarded as reasonably valid and selective, at least for the system orders shown. In each case, it will be noted that the optimum n^{th}-order system response (where $n > 2$) can be fairly well approximated by adding a time lag to the optimum second-order response. This time lag is conveniently chosen as equal to the difference between the delay time of the actual system and the delay time of a zero time lag second-order system. (Note that the delay time is defined as the time for the response to achieve 50 percent of its final value.)

A possible procedure for approximating to the performance measures of high-order systems now becomes apparent. The actual system may be replaced by an equivalent second-order system with time lag, and the performance measure calculated for the latter system by means of Eq 58, 59, and 60. This procedure is in fact quite simple, and of reasonable accuracy, as will be demonstrated for the ITAE performance measure. However, it does demand knowledge of the delay time, and a method for estimating this parameter will now be described.

ESTIMATION OF DELAY TIME

Each of the optimal responses shown in Fig. 15 can be roughly approximated by a delayed ramp function terminated near the first e(t) zero crossing, as sketched in Fig. 24. If the delay times of the actual and approximated response are made equal, then

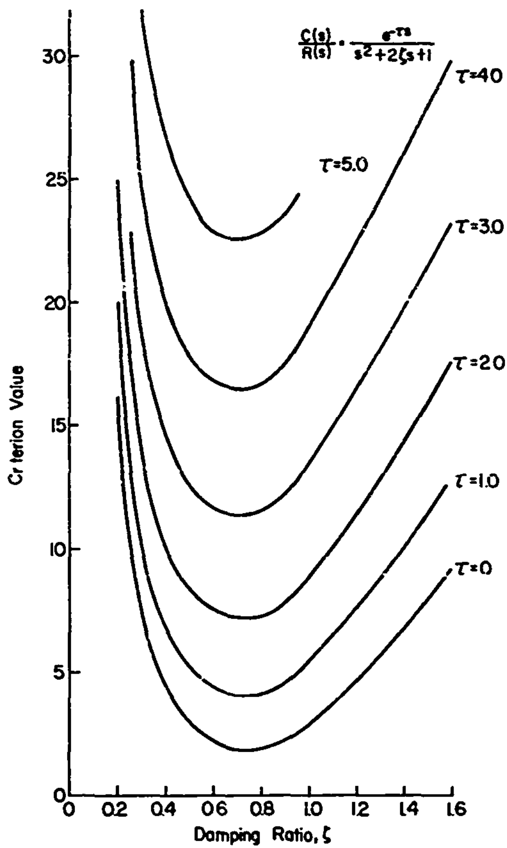


Figure 22. ITAE for a Second-Order Zero-Position-Error System with a Time Delay, τ

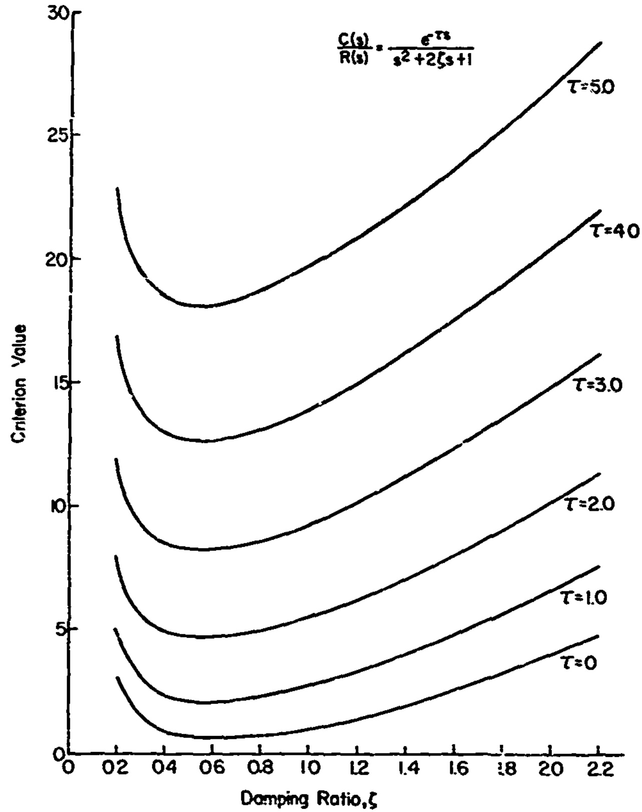


Figure 23. ITE^2 for a Second-Order Zero-Position-Error System with a Time Delay, τ

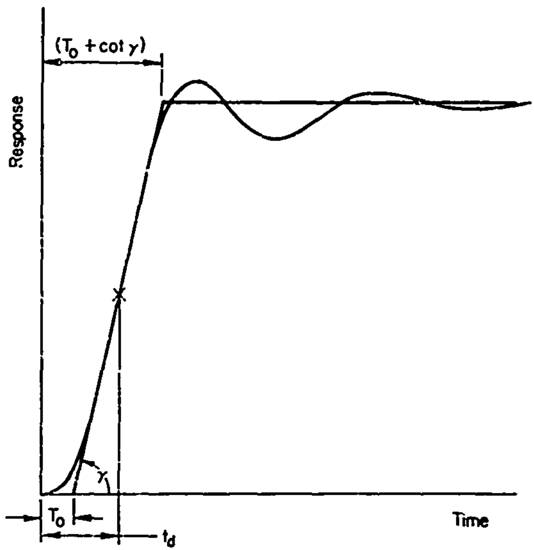


Figure 24. Ramp Approximation to Optimal System

$$t_d = T_0 + \frac{1}{2} \cot \gamma \tag{82}$$

T_0 and γ may be further related to the actual response by demanding that the integrated error of the actual and approximate responses shall be equal. This implies that

$$q_1 = \frac{(T_0 + \cot \gamma) + T_0}{2} \tag{83}$$

where q_1 is the coefficient of s in the system transfer function. (Note: q_1 = IE for a unit-numerator system) (See Chapter V) If the ramp approximation is valid (as it is for the minimum ITE^2, IT^2E^2, IT^3E^2, and ITAE systems of Fig. 15, Eq 83 will be closely satisfied, and can be combined with Eq 82 to give

$$q_1 = t_d \tag{84}$$

Figures 25 and 26 show that Eq 84 is closely satisfied for the minimum ITAE, ITE^2, IT^2E^2, and IT^3E^2 systems. It also predicts the delay time for some nonoptimal systems with good accuracy. For example, Fig. 27 illustrates the delay time/q_1 relationship for the Butterworth filters of first- through eighth-order. The indicial responses of these filters are graphed in Fig. 28. It will be observed that even for the eighth-order filter, the delay time exceeds q_1 by only 6.8 percent. It is therefore concluded that Eq. 84 predicts the delay time with good accuracy for optimum ITAE, ITE^2, IT^2E^2, IT^3E^2 systems, and for systems such as Butterworth filters which have responses approaching these optima.

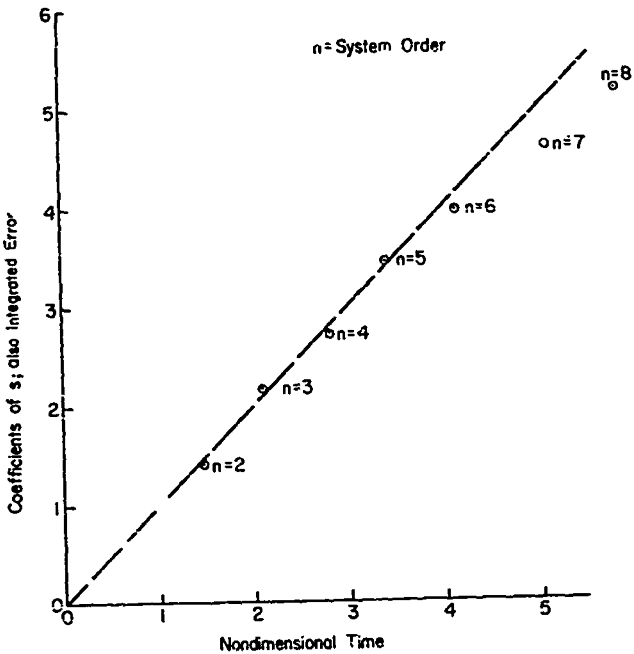
INTERPRETATION OF q_1

The coefficient q_1 is equal to the IE for a unit-numerator system, since

$$IE = \lim_{s \to 0} s \cdot \frac{E(s)}{s} = \lim_{s \to 0} \frac{s^{n-1} + q_{n-1}s^{n-2} + \cdots + q_1}{s^n + q_{n-1}s^{n-1} + \cdots + q_1 s + 1} \tag{85}$$

q_1 (which is the generalized inverse velocity constant of Ref. 72) can also be interpreted in terms of the open-loop system characteristics, as follows:

Put $$\frac{C(s)}{R(s)} = \frac{1}{s^n + q_{n-1}s^{n-1} + \cdots + q_1 s + 1} = \frac{1}{1 + \frac{1}{KG(s)}} \tag{86}$$



Delay Time (Time To 50% Final Value)

Figure 25. Variation of Delay Time with Integrated Error for Optimum ITAE Zero-Position-Error Systems

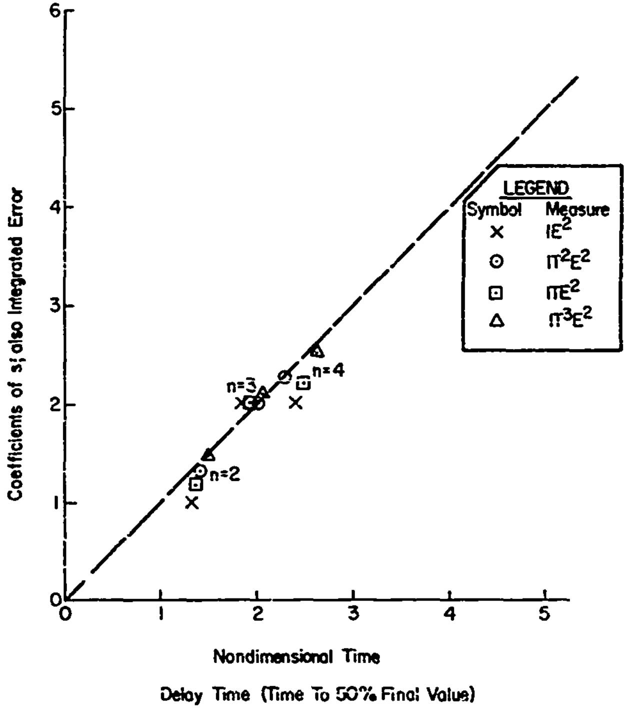


Figure 26. Variation of Delay Time with Integrated Error for Optimum IE^2, ITE^2, IT^2E^2, and IT^3E^2 Systems

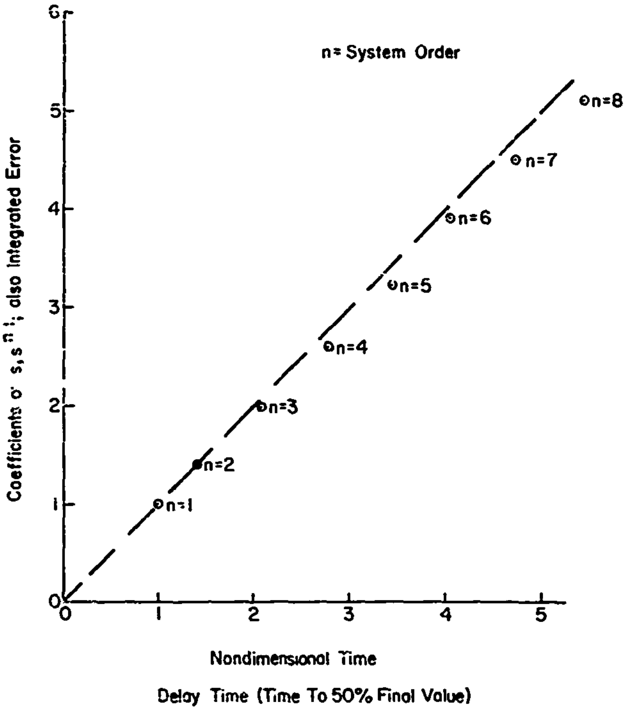


Figure 27. Variation of Delay Time with Integrated Error for Butterworth Filters

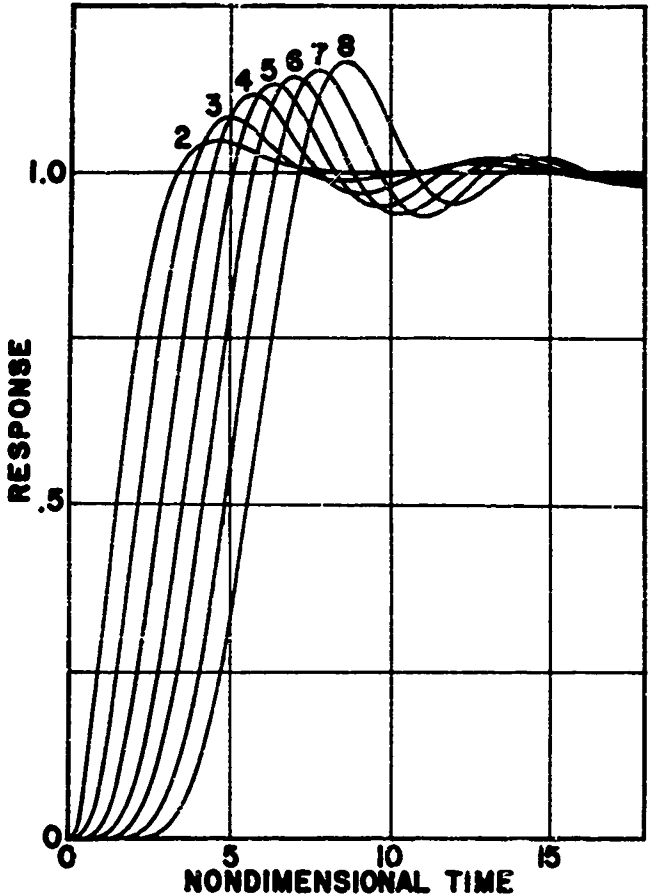


Figure 28. Indicial Responses of Butterworth Filters of First Through Eighth Orders (reproduced from Ref. 37)

then, $$\frac{1}{KG(s)} = q_1 s(\lambda_1 s + 1)(\lambda_2 s + 1) \dots (\lambda_{n-1} s + 1) \qquad (87)$$

where $-\frac{1}{\lambda_1}$, $-\frac{1}{\lambda_2}$, etc. are the open-loop roots

Because for a normalized zero-position-error system, $1/G(s)$ is always of the form $s(\lambda_1 s + 1)(\lambda_2 s + 1) \dots (\lambda_{n-1} s + 1)$,

$$\therefore \quad q_1 = \frac{1}{K} \qquad (88)$$

where K is the normalized gain (or inverse velocity constant).

A further interpretation can be obtained by applying formulae expressing the coefficients of a polynomial in terms of its roots (e.g., Ref. 74).

Let

$$s^n + q_{n-1}s^{n-1} + \dots + q_1 s + 1 = (s - \alpha_1)(s - \alpha_2) \dots (s - \alpha_n) \qquad (89)$$

where $\alpha_1 . \alpha_2 . \alpha_n$ are the closed-loop poles.

Now $$\alpha_1 \alpha_2 \alpha_3 \cdots \alpha_n = (-1)^n \qquad (90)$$

and

$$\alpha_2\alpha_3\alpha_4\cdots\alpha_n + \alpha_1\alpha_3\alpha_4\cdots\alpha_n + \alpha_1\alpha_2\alpha_4\cdots\alpha_n + \dots + \alpha_1\alpha_2\alpha_3\cdots\alpha_{n-1} = (-1)^{n-1} q_1 \qquad (91)$$

Dividing Eq 91 by Eq 90

$$\frac{1}{\alpha_1} + \frac{1}{\alpha_2} + \frac{1}{\alpha_3} + \dots \frac{1}{\alpha_n} = -q_1 \qquad (92)$$

Hence, for $\alpha_{1,2 \dots n}$ all real, q_1 is simply equal to the sum of the system time constants. (A second-order factor yields two 'time constants' summing to the appropriate $2\zeta/\omega_n$)

Some further approximations to delay time are discussed later in this chapter.

Having obtained a good approximation to delay time, and interpreted its significance, the result may now be applied to the calculation of performance measures.

EXAMPLE OF APPROXIMATE CALCULATION OF PERFORMANCE MEASURES FOR HIGH-ORDER SYSTEMS

To demonstrate the method proposed, it has been applied to the calculation of ITAE for normalized unit-numerator systems of third- through eighth-order. The optimum response of each of these systems has been approximated by a delayed second-order system with a damping ratio of 0.7. Figure 29 shows how the ITAE of such a system varies with the time lag. The measured* and predicted optimum ITAE are compared in Fig. 30. The agreement is seen to be good, especially for systems of third- through sixth-order. The slight falloff in accuracy obtained with seventh- and eighth-order systems is due to the departure of these systems from the approximating form. Figure 31 illustrates this well; the egregious behavior of the seventh- and eighth-order time histories is matched by a correspondingly unexpected distribution of poles (Fig. 21 of Ref. 37). There does not seem to be any explanation of this phenomenon. In this connection, the departure of the delay time from the predicted value for the seventh- and eighth-order systems shown in Fig. 25 should also be noted. For high-order systems, it is important that the effective τ should be estimated as closely as possible. Thus, wherever exact values of the delay time are available, they should be used in preference to the approximate $\tau_d \doteq q_1$. This approximation is satisfactory for systems of third- through sixth-order, but underestimates ITAE for the eighth-order system by 20 percent. However, in view of the large overshoots of the "optimum" ITAE seventh- and eighth-order systems, it is doubtful whether this discrepancy will be significant in practical optimization problems.

FURTHER APPROXIMATIONS TO THE DELAY TIME

In the course of developing the approximation described above, a number of alternatives were investigated. It is felt that a brief discussion of two of these would be of interest.

---

*The measured ITAE values for the sixth-, seventh-, and eighth-order systems were obtained by applying Simpson's rule and the trapezoidal integration rule to the optimum responses shown in Fig. 31, and by then averaging the results. Measured ITAE values for optimum second- through fifth-order systems are given in Ref. 37.

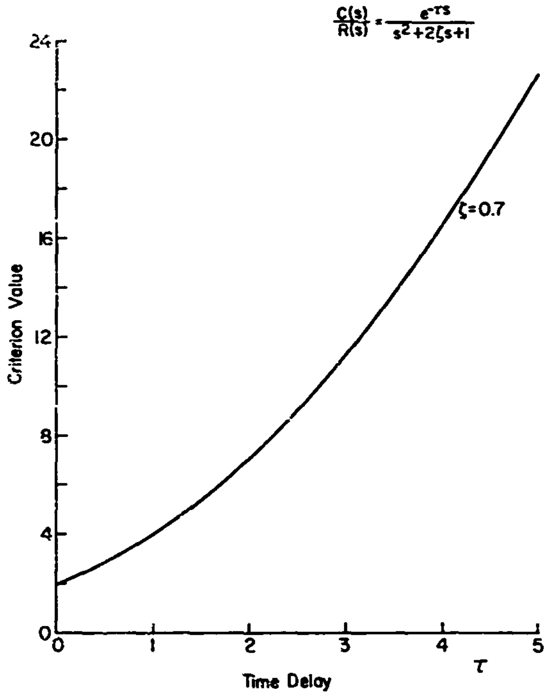


Figure 29. Effect of a Time Delay on ITAE for a Unit-Numerator Second-Order System

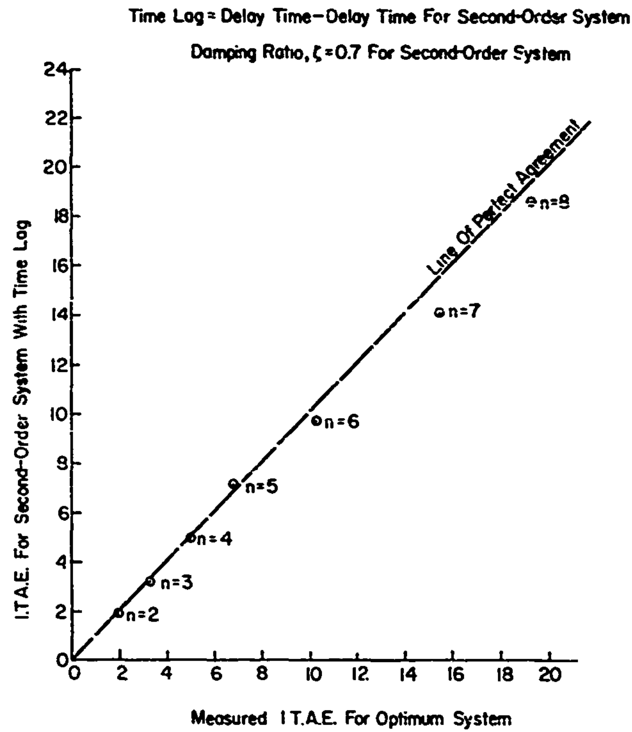


Figure 30. Comparison of Measured and Predicted ITAE

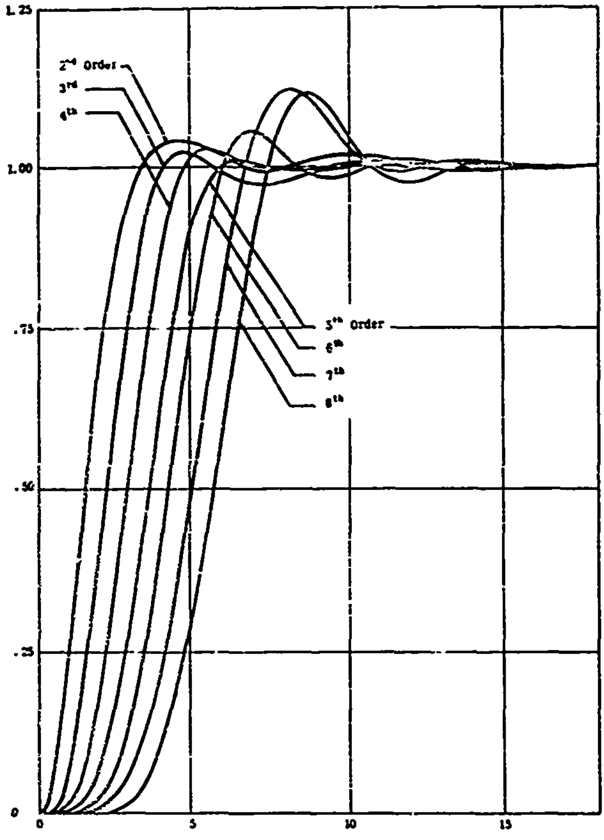


Figure 31. Indicial Responses of ITAE Standard Forms of Second- Through Eighth-Order (reproduced from Ref. 37)

First, delay time was plotted against the coefficient of s^{n-1} for various optimal systems as shown in Fig. 32. The relation is not linear, except for the binomial filters where the coefficients of s and s^{n-1} are identical.

As shown in Fig. 33, a fairly linear relation exists for the optimum ITAE systems between delay time and system order; this linearity is retained for binomial filters although the slope changes. Linearity for the binomial filters would be anticipated since the coefficient of s^{n-1} (= coefficient of s) is equal to the system order. For the optimum ITE^2, IT^2E^2, and IT^3E^2 systems, the correlation with system order is not as good.

It is concluded that both of these approximations yield lower accuracy than the relationship expressed by Eq 84.

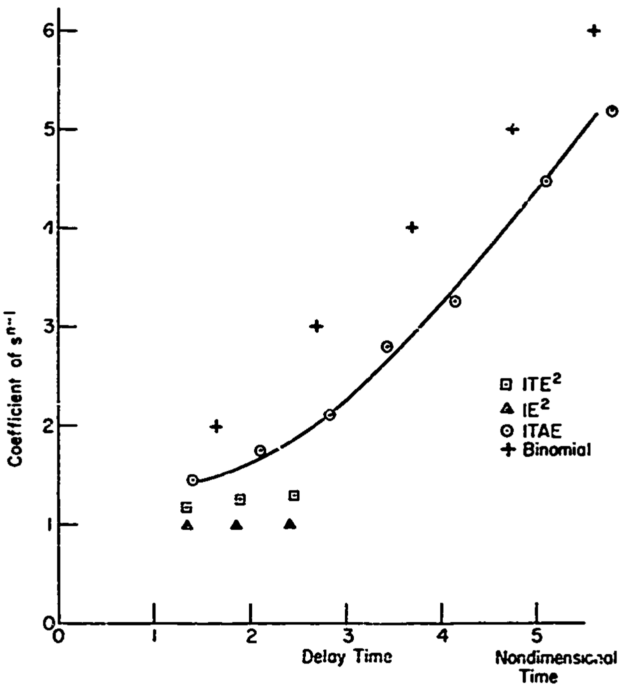


Figure 32. Correlation of Delay Time with Coefficient of s^{n-1} for n^{th}-Order Unit-Numerator ITAE Standard Forms

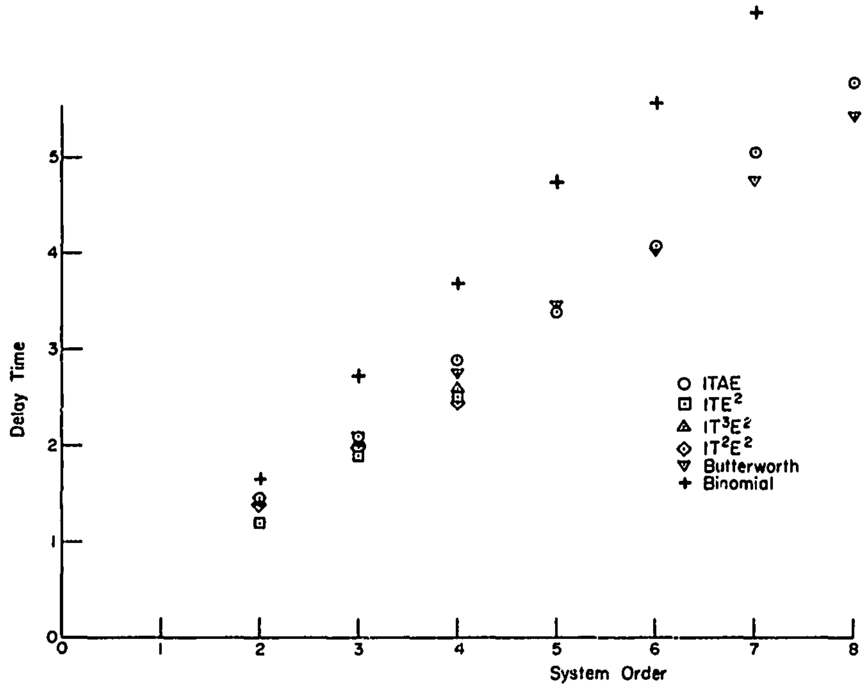


Figure 33. Correlation of Delay Time with System Order for Normalized Unit-Numerator Standard Forms

CHAPTER V

PERFORMANCE MEASURES FOR GENERAL DETERMINISTIC INPUTS

The step input has become standardized as the input required to generate the responses from which most performance measures are obtained. Indicial responses are easily generated on an analog computer, and provide a convenient means to calculate the response to more general inputs through the use of Duhamel's integral, or equivalent time-series procedures. It would be extremely convenient if performance measures appropriate to inputs other than steps could be calculated in a similarly direct fashion. (In general, it is necessary first to calculate the response, and then to apply the measure to the response). In this chapter it is shown how for some measures this procedure can be "short-circuited" by calculating the response measure from measures applied to the input and to the system indicial or impulsive response. Thus, instead of applying a measure to a Laplace transform of high order, it may be calculated by algebraic operations on measures applied to two Laplace transforms of lower order. It is also shown how certain performance measures of closed-loop systems may similarly be calculated directly from performance measures for the open-loop system.

These procedures constitute the first step towards the establishment of a "calculus of performance measures," i.e., a method of expressing performance measures for complicated systems in terms of (more readily computed) measures for simpler systems. This chapter concludes with a brief discussion of error coefficients.

IE FOR GENERAL INPUTS

The response to a unit impulse, or Dirac δ function, will be referred to here as the "impulsive admittance." It is also known as the "weighting function" and "memory function." A system having an impulsive admittance which is zero at $t \rightarrow \infty$ will now be considered. By the final value theorem

$$\lim_{s \rightarrow 0} s \cdot \frac{E(s)}{R(s)} = 0 \qquad (93)$$

where $\frac{E(s)}{R(s)}$ is the system transfer function.

The area enclosed by the impulsive admittance and the t-axis is

$$\lim_{t_1 \to \infty} \int_0^{t_1} E(t)dt = \lim_{s \to 0} s \cdot \frac{1}{s} \cdot \left.\frac{E(s)}{R(s)}\right|_{R(s) = 1} \tag{94}$$

$$= \lim_{s \to 0} E(s) = IE_{admittance} \tag{95}$$

The area enclosed by the input and the t-axis is, similarly,

$$\lim_{s \to 0} R(s) = IE_{input} \tag{96}$$

The area enclosed by the error response to a general input $r(t)$, satisfying the condition that $\lim_{t \to \infty} r(t) = 0$, is

$$\lim_{s \to 0} s \cdot \frac{1}{s} \cdot \left.\frac{E(s)}{R(s)}\right|_{R(s)=1} \cdot R(s) = \lim_{s \to 0} R(s) \times \lim_{s \to 0} \left.\frac{E(s)}{R(s)}\right|_{R(s) = 1} \tag{97}$$

This area is the integrated error response produced by the actual input. Denoting this area by IE yields the simple formula

$$IE = IE_{admittance} \times IE_{input} \tag{98}$$

The implications of this formula for system optimization using the IE criterion are worth of comment. The optimization procedure is assumed to consist of the adjustment of system parameters so that the IE is minimized,* the input being fixed. Minimization of the $IE_{admittance}$ will thus result in the minimization of the IE for any specified finite input.

Note that the conditions that have been imposed on the admittance and on the input result in a finite (or zero) IE.

*As noted in Table IV, constraints must be imposed to avoid selecting $\zeta = 0$ for minimum IE.

Integrated Error (IE) for Step Inputs

There is no essential difficulty in extending the analysis to deal with step inputs to zero-position-, velocity-, or acceleration-error systems, but the straightforward geometric interpretation of Eq 98 is lost. For example, consider a zero-position-error system having the transfer function

$$\frac{C(s)}{R(s)} = \frac{1}{s^n + q_{n-1}s^{n-1} + \dots + q_1 s + 1} \tag{99}$$

$$\frac{E(s)}{R(s)} = \frac{s^n + q_{n-1}s^{n-1} + \dots + q_1 s}{s^n + q_{n-1}s^{n-1} + \dots + q_1 s + 1} \tag{100}$$

For a step input

$$\begin{aligned} IE &= \lim_{s \to 0} s \cdot \frac{1}{s} \cdot \frac{s^n + q_{n-1}s^{n-1} + \dots + q_1 s}{s^n + q_{n-1}s^{n-1} + \dots + q_1 s + 1} \cdot \frac{1}{s} \\ &= q_1 \end{aligned} \tag{101}$$

To apply Eq 98 here, it is necessary to replace the actual system by a substitute which has an impulsive admittance identical to the indicial response of the original system. For such a system, the transfer function is

$$\frac{E(s)}{R(s)} = \frac{s^n + q_{n-1}s^{n-1} + \dots + q_1 s}{s^{n+1} + q_{n-1}s^n + \dots + q_1 s^2 + s} \tag{102}$$

The $IE_{admittance}$ is

$$\lim_{s\to o} s \cdot \frac{1}{s} \cdot \frac{s^n + q_{n-1}s^{n-1} + \dots + q_1 s}{s^{n+1} + q_{n-1}s^n + \dots + q_1 s^2 + s} = q_1 \tag{103}$$

The IE_{input} is unity for a Dirac δ function. Hence Eq 98 reduces to the trivial form

$$IE = IE_{admittance\ of\ substitute\ system} \times 1 \tag{104}$$

Integrated Time Moment of Error (ITE)

With the same restrictions on the input and admittance as in the previous sections,

$$ITE = \lim_{t_1\to\infty} \int_o^{t_1} tE(t)dt = \lim_{s\to o} s \cdot \frac{1}{s}\left[-\frac{d}{ds}E(s)\right] = \lim_{s\to o}\left[-\frac{d}{ds}E(s)\right] \tag{105}$$

$$E(s) = \left.\frac{E(s)}{R(s)}\right._{R(s)=1} \cdot R(s) \tag{106}$$

$$\lim_{s\to o}\left[-\frac{d}{ds}E(s)\right] = \lim_{s\to o}\left\{\left[\frac{E(s)}{R(s)}\right]_{R(s)=1}\left[-\frac{d}{ds}R(s)\right]\right\}$$

$$+ \lim_{s\to o}\left\{R(s)\left(-\frac{d}{ds}\left[\frac{E(s)}{R(s)}\right]_{R(s)=1}\right)\right\} \tag{107}$$

$$=\left(\lim_{s\to o}\left[\frac{E(s)}{R(s)}\right]_{R(s)=1}\right)\left(\lim_{s\to o} -\frac{d}{ds}R(s)\right)$$

$$+\left(\lim_{s\to o} R(s)\right)\left(\lim_{s\to o} -\frac{d}{ds}\left[\frac{E(s)}{R(s)}\right]_{R(s)=1}\right) \tag{108}$$

But $$\lim_{t_1 \to \infty} \int_0^{t_1} tR(t)dt = \lim_{s \to 0} s \cdot \frac{1}{s} \left(- \frac{d}{ds} R(s) \right) = ITE_{input} \tag{109}$$

and $$\lim_{t_1 \to \infty} \int_0^{t_1} t \mathcal{L}^{-1} \frac{E(s)}{R(s)} dt =$$

$$\lim_{s \to 0} s \cdot \frac{1}{s} \left(- \frac{d}{ds} \frac{E(s)}{R(s)}_{R(s)=1} \right) = ITE_{admittance} \tag{110}$$

Hence, Eq 108 can be interpreted as

$$ITE = IE_{admittance} \times ITE_{input} + ITE_{admittance} \times IE_{input} \tag{111}$$

An example of the calculation of the ITE by the use of this formula now follows. The implications of Eq 111 with regard to optimization are discussed subsequently.

Example of the Calculation of ITE for a Second-Order System

The following example illustrates the procedure for calculating ITE using Eq 111, and checks the result against that obtained by direct calculation.

The transfer function of the system considered is

$$\frac{E(s)}{R(r)} = \frac{1}{s + a} - \frac{1}{s + b} \tag{112}$$

The input is assumed to be described by $F(t) = e^{-ct} - e^{-dt}$. Evaluating each quantity on the right side of Eq 111

$$IE_{admittance} = \lim_{t_1 \to \infty} \int_0^{t_1} (e^{-at} - e^{-bt})dt \tag{113}$$

$$IE_{admittance} = \lim_{s \to o} s \cdot \frac{1}{s} \cdot \left(\frac{1}{s + a} - \frac{1}{s + b}\right) \tag{114}$$

$$= \frac{1}{a} - \frac{1}{b} \tag{115}$$

Similarly,
$$IE_{input} = \frac{1}{c} - \frac{1}{d} \tag{116}$$

$$ITE_{admittance} = \lim_{t_1 \to \infty} \int_o^{t_1} (te^{-at} - te^{-bt})dt \tag{117}$$

$$= \lim_{s \to o} s \cdot \frac{1}{s} \left[- \frac{d}{ds} \cdot \left(\frac{1}{s + a}\right) + \frac{d}{ds} \cdot \left(\frac{1}{s + b}\right)\right] \tag{118}$$

$$= -\left(\frac{1}{b^2} - \frac{1}{a^2}\right) \tag{119}$$

Similarly,
$$ITE_{input} = -\left(\frac{1}{d^2} - \frac{1}{c^2}\right) \tag{120}$$

Inserting these results in the general formula, Eq 111, yields

$$ITE = -\left(\frac{1}{a} - \frac{1}{b}\right)\left(\frac{1}{d^2} - \frac{1}{c^2}\right) - \left(\frac{1}{c} - \frac{1}{d}\right)\left(\frac{1}{b^2} - \frac{1}{a^2}\right) \tag{121}$$

$$= -\left(\frac{1}{ad^2} - \frac{1}{ac^2} - \frac{1}{bd^2} + \frac{1}{bc^2} + \frac{1}{b^2c} - \frac{1}{a^2c} - \frac{1}{db^2} + \frac{1}{a^2d}\right) \tag{122}$$

Conventionally, this formula can be derived by direct application of the Laplace transform

$$ITE = \lim_{t_1 \to \infty} \int_o^{t_1} t \left[\mathcal{L}^{-1} \left(\frac{1}{s + a} - \frac{1}{s + b}\right)\left(\frac{1}{s + c} - \frac{1}{s + d}\right)\right] dt \tag{123}$$

$$IIE = \lim_{s \to 0} s \cdot \frac{1}{s} \left[- \frac{d}{ds} \left(\frac{1}{s+a} - \frac{1}{s+b} \right) \left(\frac{1}{s+c} - \frac{1}{s+d} \right) \right] \tag{124}$$

$$\begin{aligned} ITE = \lim_{s \to 0} \Bigg[& - \frac{1}{(s+a)(s+c)^2} - \frac{1}{(s+c)(s+a)^2} + \frac{1}{(s+a)(s+d)^2} \\ & - \frac{1}{(s+d)(s+a)^2} + \frac{1}{(s+b)(s+c)^2} + \frac{1}{(s+c)(s+b)^2} \\ & - \frac{1}{(s+b)(s+d)^2} - \frac{1}{(s+d)(s+b)^2} \Bigg] \end{aligned} \tag{125}$$

$$= - \left(- \frac{1}{ac^2} - \frac{1}{a^2c} + \frac{1}{ad^2} + \frac{1}{a^2d} + \frac{1}{bc^2} + \frac{1}{b^2c} - \frac{1}{bd^2} - \frac{1}{b^2d} \right) \tag{126}$$

which is identical with the result obtained from the formula

GENERAL RELATIONS BETWEEN CERTAIN MEASURES OF OPEN-LOOP AND CLOSED-LOOP SYSTEMS

The relations of the previous section have been implicitly expressed in terms of closed-loop parameters. Equivalent open-loop expressions are now derived. These expressions enable the closed loop IE and ITE to be calculated without factoring (or even writing) the closed-loop transfer function.

Integrated Error

Denoting the open-loop transfer function by $G(s)$, the closed-loop transfer function, with unity feedback, relating error to input is

$$\frac{E(s)}{R(s)} = \frac{1}{1 + G(s)} \tag{127}$$

From the previous section

$$IE_{\text{open-loop admittance}} = G(0) \tag{128}$$

Similarly,

$$IE_{\text{closed-loop admittance}} = \lim_{s \to 0} s \cdot \frac{1}{s} \cdot \frac{1}{1 + G(s)} = \frac{1}{1 + G(0)} \qquad (129)$$

But, as shown in the previous section,

$$IE = IE_{\text{input}} \times IE_{\text{admittance}} \qquad (98)$$

so

$$IE = IE_{\text{input}} \times \frac{1}{1 + IE_{\text{open-loop admittance}}} \qquad (130)$$

and

$$IE_{\text{closed loop admittance}} = \frac{1}{1 + IE_{\text{open loop admittance}}} \qquad (131)$$

<u>Integrated Time-Moment of Error</u>

For a unity feedback closed-loop system,

$$E(s)\left[1 + G(s)\right] = R(s) \qquad (132)$$

Differentiating with respect to s,

$$E(s) \cdot \frac{d}{ds} G(s) + \left[1 + G(s)\right] \frac{d}{ds} E(s) = \frac{d}{ds} R(s) \qquad (133)$$

Taking the limit as $s \to 0$, and making use of the results previously obtained for IE and ITE, Eq 133 can be rewritten as

$$- IE \times ITE_{\text{open-loop admittance}} - ITE$$

$$- IE_{\text{open-loop admittance}} \times ITE = - ITE_{\text{input}} \qquad (134)$$

from which

$$ITE = \frac{ITE_{input} - IE \times ITE_{open\text{-}loop\ admittance}}{1 + IE_{open\text{-}loop\ admittance}} \qquad (135)$$

As noted in Chapter IV, the overshoot for optimal systems is small, and ITAE and IAE can be approximated with good accuracy by ITE and IE. Within the limits of validity of this approximation, the formulae developed in the present chapter for ITE and IE with general inputs, and for open-loop and closed-loop forms of ITE and IE, can also be applied to ITAE and IAE. Corresponding relationships for IE^2, ITE^2, and exact formulae for ITAE and IAE have not yet been obtained.

The time-weighted integrals of the impulsive response, referred to as $IT^nK_{admittance}$ in this chapter, are simply related to the dynamic error coefficients discussed briefly in Chapter II. The relationship will now be demonstrated (following Ref. 11 and 72), and its implications studied.

<u>Error Coefficients</u>

The generalized dynamic error coefficients are defined in Ref. 72 as successive coefficients of a power series expansion of $\frac{E(s)}{R(s)}$

$$\frac{E(s)}{R(s)} = E_0 + E_1 s + E_2 s^2 + \ldots \qquad (136)$$

where E_0, E_1, ... are the dynamic error coefficients.

Hence

$$E(s) = E_0 R(s) + E_1 s R(s) + E_2 s^2 R(s) + \ldots \qquad (137)$$

For an impulsive input $R(s) = 1$.

$$E(s) = E_0 + E_1 s + E_2 s^2 + \ldots \qquad (138)$$

$$\int_0^\infty e(t)dt = \lim_{s \to 0} s \cdot \frac{1}{s} E(s) = E_0 \qquad (139)$$

$$\int_0^\infty te(t)dt = \lim_{s \to 0} s \cdot \frac{1}{s} \left(- \frac{d}{ds} [E(s)] \right) = - E_1 \qquad (140)$$

$$\int_0^{\infty} t^2 e(t)dt = \lim_{s \to 0} s \cdot \frac{1}{s} \cdot \frac{d^2}{ds^2} E(s) = + 2E_2 \qquad (141)$$

Thus, the error coefficients are directly proportional to time-weighted integrals of the impulsive response.

For a second-order unit-numerator system having the transfer function

$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

$$\frac{E(s)}{R(s)} = \frac{s^2 + 2\zeta\omega_n s}{s^2 + 2\zeta\omega_n s + \omega_n^2} \qquad (142)$$

$$E_0 = \frac{1}{\omega_n^2} \qquad (143)$$

$$\frac{d}{ds}\frac{E(s)}{R(s)} = \frac{(s^2 + 2\zeta\omega_n s + \omega_n^2)(2s + 2\zeta\omega_n) - (s^2 + 2\zeta\omega_n s)(2s + 2\zeta\omega_n)}{(s^2 + 2\zeta\omega_n s + \omega_n^2)^2} \qquad (144)$$

Taking the limit as $s \to 0$

$$E_1 = -\frac{2\zeta\omega_n^3}{\omega_n^4} = -\frac{2\zeta}{\omega_n} \qquad (145)$$

Similarly, it can be shown that

$$E_2 = \frac{2}{\omega_n^2} \qquad (146)$$

None of these error coefficients results in a satisfactory criterion for an impulsive input to the second-order system considered. As noted in Chapter II, error coefficients fail as criteria because of their inability to distinguish between positive and negative error contributions to $\int_0^{\infty} e(t)dt$, $\int_0^{\infty} te(t)dt$, etc.

CHAPTER VI

ACCURACY, SENSITIVITY, AND POWER/ENERGY DEMANDS

In Chapter I, it was stated that the dynamic performance of a control system or element is assessed by considering stability, response to desired inputs, response to unwanted inputs, accuracy, insensitivity to parameter changes, and power/energy demands. This report is principally concerned with the response to desired inputs. As previously noted, the topic of response to undesired inputs requires consideration of statistically-described inputs, and falls outside the scope of the present report. Stability measures have been summarized in Table I; the remaining characteristics of accuracy, insensitivity to parameter changes, and power/energy demands are greatly dependent upon the detailed mechanization and aerodynamic properties of the particular flight control system being assessed, and do not lend themselves readily to generalized studies. This chapter therefore presents only a brief discussion of each of these three topics from a heuristic viewpoint.

Accuracy

Because accuracy is essentially the suppression of error, it can be studied by the direct or integrated error measures discussed in previous chapters. Most of these measures have been related to zero-position-error systems. Equivalent systems are normally of this form, although the linearized model of the actual system may possess a small position error. This steady-state error is explained below.

A unity feedback system will be considered, with open-loop transfer function G(s).

$$\frac{E(s)}{R(s)} = \frac{1}{1 + G(s)} \tag{147}$$

For a step input

$$\underset{t\to\infty}{E(t)} = \lim_{s\to 0} s \cdot \frac{1}{s} \cdot \frac{1}{1 + G(s)} = \frac{1}{1 + G(0)} \tag{148}$$

Thus, the system is a true zero-position-error system only for $G(0) \to \infty$. In deriving equivalent systems, this inaccuracy is generally neglected, because for a "good" system, the open-loop amplitude ratio at low frequencies is usually high. The equivalent $\frac{\theta}{\delta_e}$ system detailed in Chapter I illustrated this well. The residual error is - 0.66 db for the exact system; this error was neglected in forming the equivalent system. In general, it is believed that this neglect is justified; however, circumstances may arise where steady-state errors of this magnitude could be significant factors in performance assessment. In such cases, the system would either be modified by adding integration to the equalization to remove trim errors, or a steady-state error would be permitted, and its magnitude considered together with other criteria in judging the over-all performance.

Insensitivity to Parameter Changes

It is desirable that the dynamic performance of a given system shall not be degraded by changes in parameters occurring either as a result of a change in flight conditions or because of discrepancies between predicted and actual component characteristics. In flight control systems, the latter problem may become particularly acute, because of the well-known uncertainties in derivative estimation. Important derivatives such as N_β and L_β are often made up of components of approximately equal magnitude and opposite sign; the relative magnitude of the over-all derivative is thus very sensitive to small changes in any of its aerodynamic components.

Formulation of the performance measure being employed in analytic terms is the first step towards mastering this situation. Usually, (as in this report) this process is carried only to the stage of defining the performance measure in terms of the transfer function pole and zero locations. The further step required is to define these root locations in terms of the aerodynamic derivatives, and autopilot gains and time constants. Reference 4 describes the approximate factorization of conventional aircraft transfer functions in such literal terms. To apply the procedures outlined in this report, it is desirable to study (by the techniques outlined in Ref. 4) the sensitivity of the equivalent system parameters to uncertainties in the basic aerodynamic and autopilot characteristics. A good understanding of the effects of small changes in parameters may also be obtained through the time vector method of Ref. 12 and 21. Further analyses of sensitivity are given in Ref. 55 and 57, which investigate the changes in the roots of the

characteristic equation resulting from small changes in its coefficients, and in Ref. 75, which uses root-locus techniques to study the sensitivity of the closed-loop roots to open-loop parameter changes.

Power/Energy Demands

As noted in Ref. 67, "if the gain in a physical system is made large enough, a point is reached at which the peak acceleration of the output response exhibited in the linear model exceeds that which may be physically obtained from the power actuator of the actual system. At this point, the linear model may cease to be a valid basis for design. Either a nonlinear mathematical model must be employed, or the design procedure must be modified so that, although based on linear theory, the possibility of saturation is recognized."

It is therefore necessary to have some check as to whether or not saturation is occurring. Frequently, this may be accomplished by examining the magnitude of the peak overshoot of the output response. The appropriate systems characteristics graphs given in Chapter II, and in Ref. 13 and 24, will be found useful for this purpose.

Power/energy demands may also be of interest as direct performance measures. In space vehicles particularly, stringent limitations must be enforced on those factors. Under these conditions, dynamic performance optimization is achieved by means of combined or constrained criteria. Combined criteria are typically of the form

$$\left[\begin{array}{c}\text{indicial error measure plus a constant times total energy measure} \\ \text{equals a minimum.}\end{array}\right]$$

Constrained criteria are merely single criteria of any of the classes discussed earlier in this report with limitations upon maximum power, torque, or total energy.

Generalized assessment of combined criteria is very difficult because, in the absence of any specific application, selection of the constant becomes completely arbitrary. However, such criteria may be very valuable for a given application. It is hoped that a discussion of the use of these criteria may be included in a subsequent report.

Procedures employed for calculating certain indicial error measures may also be employed to calculate energy requirements for specified systems. Reference 54 considers an inertia-wheel attitude control system, and shows how the complex convolution method employed to calculate TE^2 can be applied with little modification to calculate the energy expressed in stabilizing the response to a step input. Because the time history of the power required is a damped sinusoid, the IAE calculation methods of the present report could also be applied to this case to calculate $\int|P|dt$ (which equals the total energy when no provision is made for recovering the kinetic energy of the inertia wheel).

SUMMARY AND CONCLUSIONS

1. Performance measures and associated criteria for linear constant coefficient systems with deterministic inputs have been investigated, with particular reference to flight control systems. The application of performance measures has been facilitated by substituting for the actual flight control system an "equivalent" low-order linearized system having similar dynamic characteristics. This equivalent system was constructed by dividing the actual system transfer function into regions of interest defined by

$$|G(j\omega)| \gg 1, \text{ over which } \left|\frac{G(j\omega)}{1 + G(j\omega)}\right| \doteq 1$$

$$|G(j\omega)| \ll 1, \text{ over which } \left|\frac{G(j\omega)}{1 + G(j\omega)}\right| \doteq |G(j\omega)|$$

$$|G(j\omega)| \doteq 1$$

The form of $\left|\frac{G(j\omega)}{1 + G(j\omega)}\right|$ in this last region defines the dominant modes of the closed-loop system response, and can usually be closely approximated by a system of first-, second-, or third order.

2. A critical and exhaustive survey of current performance measures has been conducted. Analytic forms for ITAE, IAE, etc., are presented, and a number of errors in previously published measures have been corrected. A complete correlation has been given of crossover frequency, bandwidth, phase margin, peak frequency, magnification ratio, time-to-peak, peak overshoot, and delay time for second-order unit-numerator systems. Normalized presentations are used so that practical limitations on the time scale of the response (e.g., due to power/inertia restrictions) may be taken into account separately. It is concluded that minimum ITAE and minimum ITE^2 best satisfy the combined requirements of validity, selectivity, and ease of application. The ITAE criterion yields smooth indicial responses having little overshoot, but its analytic description is complicated. Of the other indicial error measures examined, minimum IE^2 has simple analytic forms, but it selects poor indicial responses; IT^{3}E^2 responses are as good as those selected by ITAE, but IT^{3}E^2 (and also IT^{2}E^2) analytic expressions are too

complicated for general use. ITE^2 selects moderately smooth and well-damped responses (less good than ITAE), but it possesses tractable analytic forms. Therefore, ITE^2 is recommended for analytic investigations, whereas ITAE is preferred for optimization using analog computers.

3. The integrated error-response and integrated time-weighted error response of closed-loop systems to a general deterministic input have been related to the corresponding measures of the response to the impulsive input, which in turn have been expressed in terms of open-loop parameters.

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APPENDIX A

ANALYTIC EVALUATION OF IAE, ITAE, AND IT^2AE PERFORMANCE MEASURES FOR UNIT-NUMERATOR SECOND-ORDER SYSTEMS

CALCULATION OF THE INDICIAL ERROR RESPONSE OF A UNIT-NUMERATOR SECOND-ORDER SYSTEM

The transfer function of the system considered is

$$\frac{C}{R}(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} \tag{A-1}$$

$$\frac{E}{R}(s) = 1 - \frac{C}{R}(s) = \frac{s^2 + 2\zeta\omega_n s}{s^2 + 2\zeta\omega_n s + \omega_n^2} \tag{A-2}$$

For a step input

$$E(s) = \frac{s + 2\zeta\omega_n}{s^2 + 2\zeta\omega_n s + \omega_n^2} = \frac{s + 2\zeta\omega_n}{(s + \zeta\omega_n)^2 + (\omega_n\sqrt{1 - \zeta^2})^2} \tag{A-3}$$

$$c(t) = \frac{1}{\sqrt{1 - \zeta^2}} e^{-\zeta\omega_n t} \sin\left[\omega_n\sqrt{1 - \zeta^2}\, t + \psi\right] \tag{A-4}$$

where

$$\psi = \tan^{-1}\frac{1}{\zeta}\sqrt{1 - \zeta^2} = \sin^{-1}\sqrt{1 - \zeta^2} = \cos^{-1}\zeta$$

EVALUATION OF IAE

Let

$$b(t) = \frac{1}{\sqrt{1 - \zeta^2}} \sin\left[\omega_n\sqrt{1 - \zeta^2}\, t + \psi\right] \tag{A-5}$$

$$\mathcal{L}\{b(t)\} = \frac{1}{\sqrt{1 - \zeta^2}}\int_0^\infty e^{-st}\left|\sin\left[\omega_n\sqrt{1 - \zeta^2}\, t + \psi\right]\right| dt$$

If

$$\tau = t + \frac{\psi}{\omega_n\sqrt{1 - \zeta^2}} = t + \frac{\psi}{\omega}$$

$$\mathcal{L}|b(t)| = \frac{e^{\psi s/\omega}}{\sqrt{1-\zeta^2}}\left\{\int_0^{\infty} e^{-s\tau}\,|\sin\omega\tau|\,d\tau - \int_0^{\psi/\omega} e^{-s\tau}\,|\sin\omega\tau|\,d\tau\right\}$$

$$= \frac{e^{\psi s/\omega}}{\sqrt{1-\zeta^2}}\left\{-\int_0^{\psi/\omega} + \int_0^{\pi/\omega} + \int_{2\pi/\omega}^{3\pi/\omega} + \cdots - \int_{\pi/\omega}^{2\pi/\omega} - \int_{3\pi/\omega}^{4\pi/\omega} - \cdots\right\} \qquad \text{(A-6)}$$

where the integrand (which has been omitted for brevity) is

$$e^{-s\tau}\sin\omega\tau$$

$$\int e^{-s\tau}\sin\omega\tau\,d\tau = \frac{e^{-s\tau}(-s\sin\omega\tau - \omega\cos\omega\tau)}{s^2+\omega^2} \qquad \text{(A-7)}$$

With the appropriate limits, the results of the integration are

$$\int_0^{\pi/\omega} = \frac{\omega}{s^2+\omega^2}\left\{e^{-s\pi/\omega} + 1\right\}$$

$$\int_{2\pi/\omega}^{3\pi/\omega} = \frac{\omega}{s^2+\omega^2}\left\{e^{-3\pi s/\omega} + e^{-2\pi s/\omega}\right\}$$

$$-\int_0^{\psi/\omega} = \frac{\omega}{s^2+\omega^2}\left\{e^{-s\psi/\omega}\left(\frac{s}{\omega}\sin\psi + \cos\psi\right) - 1\right\}$$

$$-\int_{\pi/\omega}^{2\pi/\omega} = \frac{\omega}{s^2+\omega^2}\left\{e^{-2\pi s/\omega} + e^{-\pi s/\omega}\right\}$$

$$-\int_{3\pi/\omega}^{4\pi/\omega} = \frac{\omega}{s^2+\omega^2}\left\{e^{-4\pi s/\omega} + e^{-3\pi s/\omega}\right\}$$

$$\cdots \qquad \cdots$$

So

$$B(s) = \frac{\omega_n e^{\psi s/\omega}}{s^2 + \omega^2}\left\{1 + 2e^{-\pi s/\omega} + 2e^{-2\pi s/\omega} + \cdots + e^{-\psi s/\omega}\left(\frac{s}{\omega_n} + \zeta\right) - 1\right\} \qquad \text{(A-8)}$$

But for Re $|\pi s/\omega| > 0$,

$$1 + 2e^{-\pi s/\omega} + 2e^{-2\pi s/\omega} + \cdots = -1 + \frac{2}{1 - e^{-\pi s/\omega}} = \frac{1 + e^{-\pi s/\omega}}{1 - e^{-\pi s/\omega}} = \coth\left(\frac{\pi s}{2\omega}\right) \qquad \text{(A-9)}$$

$$B(s) = \frac{\omega_n e^{\psi s/\omega}}{s^2 + \omega^2}\coth\frac{\pi s}{2\omega} + \frac{1}{s^2 + \omega^2}\left[s + \zeta\omega_n - \omega_n e^{\psi s/\omega}\right]$$

$$= \frac{\omega_n e^{\psi s/\omega}}{s^2 + \omega^2}\left[\coth\left(\frac{\pi s}{2\omega}\right) - 1\right] + \frac{s + \zeta\omega_n}{s^2 + \omega^2} \qquad \text{(A-10)}$$

which is the Laplace transform of $|b(t)|$. Now, because $|e(t)| = |b(t)|e^{-\zeta\omega_n t}$,

$$E_A(s) = B(s + \zeta\omega_n)$$

where $E_A(s)$ is the Laplace transform of the absolute error. Making the substitution,

$$E_A(s) = \frac{\omega_n e^{\psi(s + \zeta\omega_n)/\omega}}{(s + \zeta\omega_n)^2 + \omega^2}\left[\coth\frac{\pi(s + \zeta\omega_n)}{2\omega} - 1\right] + \frac{s + 2\zeta\omega_n}{(s + \zeta\omega_n)^2 + \omega^2} \qquad \text{(A-11)}$$

the IAE may now be found through the use of the final value theorem

$$\text{IAE} = \lim_{t\to\infty}\int_0^t |e(\tau)|d\tau = \lim_{s\to 0} s\left(\frac{E_A(s)}{s}\right)$$

$$= \frac{\omega_n e^{\psi\zeta\omega_n/\omega}}{(\zeta\omega_n)^2 + \omega^2}\left[\coth\frac{\pi\zeta\omega_n}{2\omega} - 1\right] + \frac{2\zeta\omega_n}{(\zeta\omega_n)^2 + \omega^2} \qquad \text{(A-12)}$$

Noting that

$$(\zeta\omega_n)^2 + \omega^2 = \omega_n^2$$

$$\frac{\zeta\omega_n}{\omega} = \frac{\zeta}{\sqrt{1 - \zeta^2}}$$

$$\mathrm{IAE} = \frac{e^{\frac{\zeta}{\sqrt{1-\zeta^2}}\cos^{-1}\zeta}}{\omega_n}\left[\coth\left(\frac{\pi\zeta}{2\sqrt{1-\zeta^2}}\right) - 1\right] + \frac{2\zeta}{\omega_n} \qquad \text{(A-13)}$$

EVALUATION OF ITAE

ITAE is defined as

$$\mathrm{ITAE} \quad \int_0^{\infty} t|e(t)|dt = \lim_{s \to 0}\left[s\mathcal{L}\left\{\int t|e(t)|dt\right\}\right]$$

$$= \lim_{s \to 0}\left\{s \cdot \frac{1}{s}\mathcal{L}\left[t|e(t)|\right]\right\} = \lim_{s \to 0}\left[-\frac{d}{ds}\mathcal{L}[|e(t)|]\right]$$

$$= \lim_{s \to 0}\left[-\frac{d}{ds}E_A(s)\right] \qquad \text{(A-14)}$$

where $E_A(s)$ is the Laplace transform of the absolute error (Eq A-11). $\frac{d}{ds}E_A(s)$ is calculated as follows: let

$$s + \zeta\omega_n = \alpha$$

Then Eq A-11 becomes

$$E_A(s) = \frac{1}{\alpha^2 + \omega^2}\left\{\omega_n e^{\psi\alpha/\omega}\left[\coth\frac{\pi\alpha}{2\omega} - 1\right] + \alpha + \zeta\omega_n\right\} \qquad \text{(A-15)}$$

and

$$\frac{dE_A(s)}{ds} = \frac{dE_A(s)}{d\alpha}\frac{d\alpha}{ds} = \frac{dE_A}{d\alpha} \qquad \text{(A-16)}$$

$$\frac{dE_A}{ds} = -\frac{2\alpha}{\left(\alpha^2 + \omega^2\right)^2}\left\{\omega_n e^{\psi\alpha/\omega}\left[\coth\frac{\pi\alpha}{2\omega} - 1\right] + \alpha + \zeta\omega_n\right\}$$
$$+ \frac{1}{\alpha^2 + \omega^2}\left\{\frac{\psi\omega_n}{\omega}e^{\psi\alpha/\omega}\left[\coth\frac{\pi\alpha}{2\omega} - 1\right] - \frac{\pi}{2\omega}\omega_n e^{\psi\alpha/\omega}\,\mathrm{csch}^2\frac{\pi\alpha}{2\omega} + 1\right\} \qquad \text{(A-17)}$$

$$= -\frac{1}{\alpha^2 + \omega^2}\left\{\omega_n e^{\psi\alpha/\omega}\left[\coth\frac{\pi\alpha}{2\omega} - 1\right]\left[\frac{2\alpha}{\alpha^2 + \omega^2} - \frac{\psi}{\omega} + \frac{\pi}{2\omega}\left(\coth\frac{\pi\alpha}{2\omega} + 1\right)\right]\right.$$
$$\left. + \frac{2\alpha(\alpha + \zeta\omega_n)}{\alpha^2 + \omega^2} - 1\right\} \qquad \text{(A-18)}$$

Since $\lim_{s \to 0} \alpha = \zeta\omega_n$,

$$\mathrm{ITAE} = \frac{1}{(\zeta\omega_n)^2 + \omega^2} \left\{ \omega_n e^{\psi\zeta\omega_n/\omega} \left[\coth \frac{\pi\zeta\omega_n}{2\omega} - 1 \right] \left[\frac{2\zeta\omega_n}{(\zeta\omega_n)^2 + \omega^2} - \frac{\psi}{\omega} + \frac{\pi}{2\omega} \left(\coth \frac{\pi\zeta\omega_n}{2\omega} + 1 \right) \right] + \frac{4(\zeta\omega_n)^2}{(\zeta\omega_n)^2 + \omega^2} - 1 \right\} \tag{A-19}$$

Making the substitutions

$$(\zeta\omega_n)^2 + \omega^2 = \omega_n^2 \;, \quad \frac{\zeta\omega_n}{\omega} = \frac{\zeta}{\sqrt{1 - \zeta^2}} \;, \quad \psi = \cos^{-1} \zeta$$

$$\mathrm{ITAE} = \left(\coth \frac{\pi\zeta}{2\sqrt{1 - \zeta^2}} - 1 \right) \left(4\zeta \sqrt{1 - \zeta^2} + \pi \coth \frac{\pi\zeta}{2\sqrt{1 - \zeta^2}} + 2 \sin^{-1} \zeta \right) \frac{e^{\frac{\zeta}{\sqrt{1-\zeta^2}} \cos^{-1}\zeta}}{2\omega_n^2 \sqrt{1 - \zeta^2}} + \frac{4\zeta^2 - 1}{\omega_n^2} \tag{A-20}$$

Both the IAE and ITAE have been calculated and compared with the values obtained by Graham and Lathrop (Ref. 37) (from analog computer mechanization) in Fig. 17. The analytically obtained results agree well with those of Ref. 37 for IAE and show minor discrepancies on ITAE.

As $\zeta \to 1$, the ITAE expression (Eq A-20) becomes indeterminate, but it can be shown that in the limit it reduces to $(4\zeta^2 - 1)/\omega_n^2 = 3/\omega_n^2$ for $\zeta \to 1$.

Additionally, the measures may be written in terms of the normalized measures by application of Eq 26 and 27 of the main text of this report.

$$\mathrm{IAE}(\zeta, \omega_n) = \frac{1}{\omega_n} \mathrm{IAE}(\zeta, 1) \tag{A-21}$$

$$\mathrm{ITAE}(\zeta, \omega_n) = \frac{1}{\omega_n^2} \mathrm{ITAE}(\zeta, 1) \tag{A-22}$$

CALCULATION OF IT^2AE

$$IT^2AE = \lim_{s \to 0} \frac{d^2E_A(s)}{ds^2} \tag{A-23}$$

As shown previously,

$$\frac{dE_A(s)}{ds} = -\underbrace{\frac{1}{\alpha^2+\omega^2}}_{A}\left\{\underbrace{\omega_n e^{\mp\pi/\omega}}_{B}\underbrace{\left[\coth\frac{\pi\alpha}{2\omega}-1\right]}_{C}\underbrace{\left[\frac{2\alpha}{\alpha^2+\omega^2}-\frac{\nu}{\omega}+\frac{\pi}{2\omega}\left(\coth\frac{\pi\alpha}{2\omega}+1\right)\right]}_{D}+\underbrace{\frac{2\alpha(\alpha+\zeta\omega_n)}{\alpha^2+\omega^2}-1}_{E}\right\} \tag{A-24}$$

The differentiation may be simplified by grouping the terms as shown by the brackets, and operating on a group at a time.

Then
$$\frac{dE_A}{ds} = -A(BCD + E) \tag{A-25}$$

and
$$\frac{d^2E_A(s)}{ds^2} = -ABCD\left(\frac{A'}{A}+\frac{B'}{B}+\frac{C'}{C}+\frac{D'}{D}\right) - AE\left(\frac{A'}{A}+\frac{E'}{E}\right) \tag{A-26}$$

The calculation of each group is outlined on the following pages.

$$A = \frac{1}{\alpha^2 + \omega^2}, \qquad \lim_{s \to 0} A = \frac{1}{\omega_n^2} \tag{A-27}$$

$$A' = -\frac{2\alpha}{(\alpha^2 + \omega^2)^2} \tag{A-28}$$

$$\lim_{s \to 0} \frac{A'}{A} = \frac{-2\zeta\omega_n}{\omega_n^2} = -\frac{2\zeta}{\omega_n} \tag{A-29}$$

$$B = \omega_n e^{\mp\alpha/\omega}, \qquad \lim_{s \to 0} B = \omega_n e^{\frac{\zeta}{\sqrt{1 - \zeta^2}} \cos^{-1}\zeta} \tag{A-30}$$

$$B' = \frac{\mp\omega_n}{\omega} e^{\mp\alpha/\omega} \tag{A-31}$$

$$\lim_{s \to 0} \frac{B'}{B} = \frac{\mp}{\alpha} = \frac{\cos^{-1}\zeta}{\omega_n \sqrt{1 - \zeta^2}} \tag{A-32}$$

$$C = \coth \frac{\pi\alpha}{2\omega} - 1, \qquad \lim_{s \to 0} C = \coth \frac{\pi}{2} \cdot \frac{\zeta}{\sqrt{1 - \zeta^2}} - 1 \tag{A-33}$$

$$C' = -\frac{\pi}{2\omega} \operatorname{csch}^2 \frac{\pi\alpha}{2\omega} \tag{A-34}$$

$$\lim_{s \to 0} \frac{C'}{C} = -\frac{\pi}{2\omega} \left(\coth \frac{\pi}{2} \cdot \frac{\zeta}{\sqrt{1 - \zeta^2}} + 1 \right) \tag{A-35}$$

$$D = \frac{2\alpha}{\alpha^2 + \omega^2} - \frac{1}{\alpha} + \frac{\pi}{2\omega}\left(\coth \frac{\pi\alpha}{2\omega} + 1\right) \tag{A-36}$$

$$D' = \frac{2(\omega^2 - \alpha^2)}{(\alpha^2 + \omega^2)^2} - \left(\frac{\pi}{2\omega}\right)^2 \left(\coth^2 \frac{\pi\alpha}{2\omega} - 1\right) \tag{A-37}$$

$$\lim_{s \to 0} \frac{D'}{D} = \frac{1}{\omega_n \sqrt{1 - \zeta^2}} \; \frac{2(1 - 2\zeta^2)(1 - \zeta^2) - \frac{\pi^2}{4} (\coth^2\beta - 1)}{2\zeta\sqrt{1 - \zeta^2} - \cos^{-1}\zeta + \frac{\pi}{2}(\coth \beta + 1)} \tag{A-38}$$

$$E = \frac{2\alpha(\alpha + \zeta\omega_n)}{\alpha^2 + \omega^2} - 1 \tag{A-39}$$

$$E' = \frac{2\zeta\omega_n\omega^2 + 4\alpha\omega^2 - 2\alpha^2\zeta\omega_n}{(\alpha^2 + \omega^2)^2} \tag{A-40}$$

$$\lim_{s \to 0} \frac{E'}{E} = \frac{2\zeta(3 - 4\zeta^2)}{\omega_n(4\zeta^2 - 1)} \tag{A-41}$$

$$\lim_{s \to 0} ABCD = \frac{\zeta e^{\gamma}(\coth \beta - 1)}{\omega_n^2\sqrt{1 - \zeta^2}} \left[2\sqrt{1 - \zeta^2} - \frac{\cos^{-1}\zeta}{\zeta} + \frac{\pi}{2\zeta} (\coth \beta + 1)\right] \tag{A-42}$$

where $$\gamma = \frac{\zeta \cos^{-1}\zeta}{\sqrt{1 - \zeta^2}}, \qquad \beta = \frac{\pi\zeta}{2\sqrt{1 - \zeta^2}} \tag{A-43}$$

$$\lim_{s \to 0} AE = \frac{4\zeta^2 - 1}{\omega_n^2} \tag{A-44}$$

$$
\frac{d^2E_A(s)}{ds^2} = -\frac{\zeta e^{\gamma}(\coth\beta-1)}{\omega_n^2\sqrt{1-\zeta^2}}\left[2\sqrt{1-\zeta^2}-\frac{\cos^{-1}\zeta}{\zeta}+\frac{\pi}{2\zeta}(\coth\beta+1)\right]\Bigg\{-\frac{2\zeta}{\omega_n}+\frac{\cos^{-1}\zeta}{\omega_n\sqrt{1-\zeta^2}}-\frac{\pi}{2\omega_n\sqrt{1-\zeta^2}}(\coth\beta+1)
$$

$$
+\frac{1}{\omega_n\sqrt{1-\zeta^2}}\,\frac{2(1-2\zeta^2)(1-\zeta^2)-\frac{\pi^2}{4}(\coth^2\beta-1)}{2\zeta\sqrt{1-\zeta^2}-\cos^{-1}\zeta+\frac{\pi}{2}(\coth\beta+1)}\Bigg\}-\frac{4\zeta^2-1}{\omega_n^2}\left[-\frac{2\zeta}{\omega_n}+\frac{2\zeta(3-4\zeta^2)}{\omega_n(4\zeta^2-1)}\right] \tag{A-45}
$$

$$
=\frac{\zeta e^{\gamma}(\coth\beta-1)}{\omega_n^2\sqrt{1-\zeta^2}}\Bigg\{\left[2\zeta-\frac{\cos^{-1}\zeta}{\sqrt{1-\zeta^2}}+\frac{\pi}{2\sqrt{1-\zeta^2}}(\coth\beta+1)\right]\left[2\sqrt{1-\zeta^2}-\frac{\cos^{-1}\zeta}{\zeta}+\frac{\pi}{2\zeta}(\coth\beta+1)\right]
$$

$$
-\frac{1}{\zeta\sqrt{1-\zeta^2}}\left[2(1-2\zeta^2)(1-\zeta^2)-\frac{\pi^2}{4}(\coth^2\beta-1)\right]\Bigg\}+\frac{8\zeta(2\zeta^2-1)}{\omega_n^3} \tag{A-46}
$$

$$
=\frac{\zeta e^{\gamma}(\coth\beta-1)}{\omega_n^2\sqrt{1-\zeta^2}}\Bigg\{4\zeta\sqrt{1-\zeta^2}-2\cos^{-1}\zeta+\pi\coth\beta+\pi-2\cos^{-1}\zeta+\frac{(\cos^{-1}\zeta)^2}{\zeta\sqrt{1-\zeta^2}}-\frac{\pi\cos^{-1}\zeta}{2\zeta\sqrt{1-\zeta^2}}\coth\beta
$$

$$
-\frac{\pi\cos^{-1}\zeta}{2\zeta\sqrt{1-\zeta^2}}+\pi\coth\beta+\pi-\frac{\pi\cos^{-1}\zeta}{2\zeta\sqrt{1-\zeta^2}}\coth\beta-\frac{\pi\cos^{-1}\zeta}{2\zeta\sqrt{1-\zeta^2}}+\frac{\pi^2}{4\zeta\sqrt{1-\zeta^2}}(\coth\beta+1)^2-\frac{2(1-2\zeta^2)\sqrt{1-\zeta^2}}{\zeta}
$$

$$
+\frac{\pi^2}{4\zeta\sqrt{1-\zeta^2}}\coth^2\beta-\frac{\pi^2}{4\zeta\sqrt{1-\zeta^2}}\Bigg\}+\frac{8\zeta(2\zeta^2-1)}{\omega_n^3} \tag{A-47}
$$

$$
s^2\Delta E=\frac{\zeta}{\omega_n^2\sqrt{1-\zeta^2}}\left(\coth\frac{\pi\zeta}{2\sqrt{1-\zeta^2}}+1\right)e^{\frac{\zeta}{\sqrt{1-\zeta^2}}\cos^{-1}\zeta}\Bigg\{\frac{2\sqrt{1-\zeta^2}}{\zeta}(4\zeta^2-1)+\left[\frac{\cos^{-1}\zeta-\pi}{\zeta\sqrt{1-\zeta^2}}-4\right]\cos^{-1}\zeta+2\pi
$$

$$
+\pi\coth\frac{\pi\zeta}{2\sqrt{1-\zeta^2}}\left[2-\frac{\cos^{-1}\zeta}{\zeta\sqrt{1-\zeta^2}}+\frac{\pi\coth\frac{\pi\zeta}{2\sqrt{1-\zeta^2}}}{2\zeta\sqrt{1-\zeta^2}}+\frac{\pi}{2\zeta\sqrt{1-\zeta^2}}\right]\Bigg\}+\frac{8\zeta(2\zeta^2-1)}{\omega_n^3} \tag{A-48}
$$

The last term of Eq A-48 is the IT^2E,

$$IT^2E = \frac{8\zeta(2\zeta^2 - 1)}{\omega_n^3} \tag{A-49}$$

The normalized IT^2AE and IT^2E are graphed in Fig. 18, and compared with the values obtained by Graham and Lathrop (Ref. 57). A scaling error of 2 is detected in the latter curve.

APPENDIX B

A METHOD FOR EVALUATING IAE AND ITAE FOR THIRD-ORDER SYSTEMS

A METHOD FOR EVALUATING IAE AND ITAE FOR THIRD-ORDER SYSTEMS

This appendix presents an analytic procedure for evaluating IAE and ITAE for third-order systems. The zeros of the error time history for these systems are not equally spaced, and the procedure in Appendix A cannot be used. In the work presented here, the error time history is described by a Fourier-like series, the coefficients of which are time-dependent. A similar technique was employed in Ref. 20 to describe the output of a linear full-wave rectifier subjected to a damped sinusoidal input (second-order error response). This appendix extends the procedure of Ref. 20 to third-order responses consisting of a damped sinusoid plus an exponential term, and shows how the resulting expression for $|e(t)|$ may be integrated to give IAE. The procedure for obtaining ITAE is also outlined.

To examine the accuracy of the method (i.e., the number of harmonics required to give acceptable representations of the actual functions), several examples of the calculation of $|e(t)|$ are given. It is shown that only three or four harmonics need be taken in most cases to achieve an accuracy of within 2 or 3 percent. The convergence of the IAE series is even more rapid.

DETERMINATION OF THE TRANSFER FUNCTION OF A FULL-WAVE RECTIFIER

A full-wave linear rectifier has a transfer characteristic shown in Fig. 34.

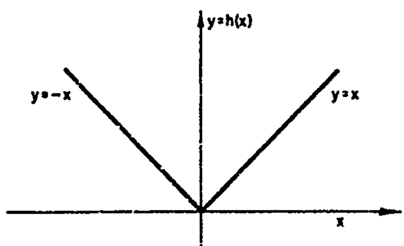

Figure 34. Rectifier Transfer Characteristic

where x is the input signal.

Defining

$$h_+(x) = \begin{cases} x & \text{when} \quad x > 0 \\ 0 & \text{when} \quad x \leq 0 \end{cases}$$

$$h_-(x) = \begin{cases} 0 & \text{when} \quad x \geq 0 \\ -x & \text{when} \quad x < 0 \end{cases} \tag{B-1}$$

Then

$$y = h(x) = h_+(x) + h_-(x) \tag{B-2}$$

$h_+(x)$ and $h_-(x)$ have unilateral Laplace transforms, $f_+(\omega)$ and $f_-(\omega)$, respectively, where

$$f_+(\omega) = \int_0^\infty h_+(x)\, e^{-\omega x}\, dx$$

$$f_-(\omega) = \int_{-\infty}^0 h_-(x)\, e^{-\omega x}\, dx \tag{B-3}$$

The output of the full-wave linear rectifier is given by the following inverse Laplace transform (Fig. 35):

$$h(x) = \frac{1}{2\pi j} \int_{\epsilon - j\infty}^{\epsilon + j\infty} f_+(\omega)\, e^{\omega x}\, d\omega + \frac{1}{2\pi j} \int_{-\epsilon - j\infty}^{-\epsilon + j\infty} f_-(\omega)\, e^{\omega x}\, dx \tag{B-4}$$

Where $\epsilon > 0$

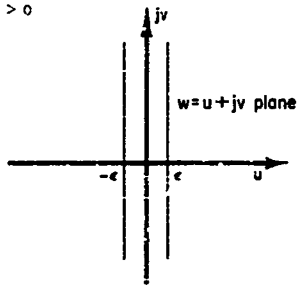

Figure 35. Inversion Contours

The transforms $f_+(\omega)$ and $f_-(\omega)$ can be evaluated as follows:

$$
\begin{aligned}
f_+(\omega) &= \int_0^{\infty} x\, e^{-\omega x} dx = \frac{1}{\omega^2} \\
f_-(\omega) &= \int_{-\infty}^{0} (-x) e^{-\omega x} dx = \int_0^{\infty} x\, e^{-(-\omega)x} dx = \frac{1}{(-\omega)^2}
\end{aligned}
\tag{B-5}
$$

and $h(x)$ in Eq B-2 becomes

$$
h(x) = \frac{1}{2\pi j} \int_{c-j\infty}^{c+j\infty} \frac{1}{\omega^2} e^{\omega x} d\omega + \frac{1}{2\pi j} \int_{-c-j\infty}^{-c+j\infty} \frac{1}{(-\omega)^2} e^{\omega x} d\omega
$$

which can be condensed to give

$$
h(x) = \frac{1}{2\pi j} \int_{c-j\infty}^{c+j\infty} \left(\frac{e^{\omega x} + e^{-\omega x}}{\omega^2} \right) d\omega \tag{B-6}
$$

CHARACTERIZATION OF THE INPUT SIGNAL, $x(t)$

The input signal is the indicial error response of a normalized unit-numerator third-order system

$$
\frac{C(s)}{R(s)} = \frac{1}{s^3 + b_o s^2 + c_o s + 1} \tag{B-7}
$$

for $R(s) = 1/s$

$$
E(s) = \frac{s^2 + b_o s + c_o}{s^3 + b_o s^2 + c_o s + 1} = \frac{s^2 + b_o s + c_o}{(s + \gamma)\left[(s + \alpha)^2 + \beta^2\right]} \tag{B-8}
$$

where

$$
\begin{aligned}
(\alpha^2 + \beta^2)\gamma &= 1 \\
b_o &= 2\alpha + \gamma \\
c_o &= \frac{1 + 2\alpha\gamma^2}{\gamma}
\end{aligned}
$$

from which

$$e(t) = ae^{-\gamma t} + ke^{-\alpha t} \sin(\beta t + \psi) \tag{B-9}$$

where

$$a = \frac{1}{\gamma\left[(\alpha - \gamma)^2 + \beta^2\right]}$$

$$k = \frac{\gamma}{\beta} a^{1/2}$$

$$\psi = \tan^{-1}\frac{\beta}{\alpha} - \tan^{-1}\frac{\beta}{\gamma - \alpha}$$

e(t) may be written

$$x(t) = e(t) = A(t) + V(t) \cos \theta(t) \tag{B-10}$$

where

$$A(t) = ae^{-\gamma t}$$

$$V(t) = ke^{-\alpha t}$$

$$\theta(t) = \beta t + \psi - \frac{\pi}{2}$$

CHARACTERIZATION OF THE OUTPUT SIGNAL

The output may be obtained by substituting $x(t)$(Eq B-10) in Eq B-6.

$$y(t) = \frac{1}{2\pi j} \int_{\epsilon - j\infty}^{\epsilon + j\infty} \frac{e^{\omega(A + V\cos\theta)} + e^{-\omega(A + V\cos\theta)}}{\omega^2} \, d\omega \tag{B-11}$$

The exponential terms may be expanded using the following uniformly convergent series (Jacobi-Anger formula, Ref. 52, p. 18 and Ref. 20, p. 282).

$$e^{z \cos \theta} = \sum_{m=0}^{\infty} \mathcal{E}_m I_m(z) \cos \theta \tag{B-12}$$

where $\mathcal{E}_m = 2 \; (m = 1, 2, 3, \cdots)$

$\mathcal{E}_0 = 1$

$I_m(z)$ is the modified Bessel function of the first kind.

Making use of the relationship $I_m(-z) = (-1)^m I(z)$,

$$y(t) = \sum_{m=0}^{\infty} \frac{\varepsilon_m \cos m\theta}{2\pi j} \int_{\epsilon-j\infty}^{\epsilon+j\infty} \frac{I_m(\omega V)\left[e^{\omega A} + (-1)^m e^{-\omega A}\right]d\omega}{\omega^2} \tag{B-13}$$

Now let

$$\zeta = \omega V \qquad b(t) = \frac{A(t)}{V(t)} \tag{B-14}$$
$$\delta = \epsilon V$$

$$y(t) = \sum_{m=0}^{\infty} \frac{V(t)\varepsilon_m \cos m\theta}{2\pi j} \int_{\delta-j\infty}^{\delta+j\infty} \frac{I_m(\zeta)\left[e^{b\zeta} + (-1)^m e^{-b\zeta}\right]d\zeta}{\zeta^2}$$

$$y(t) = V(t) \sum_{m=0}^{\infty} \varepsilon_m C(t,m) \cos m\theta \tag{B-15}$$

where

$$C(t,m) = \frac{1}{2\pi j} \int_{\delta-j\infty}^{\delta+j\infty} \frac{I_m(\zeta)\left[e^{\zeta b(t)} + (-1)^m e^{-\zeta b(t)}\right]d\zeta}{\zeta^2}$$

Because (from Eq B-2 and Eq B-10)

$$y(t) = |e(t)| = |A(t) + V(t)\cos\theta|$$
$$y(t) = V(t)\,|b(t) + \cos\theta(t)| \tag{B-16}$$

In effect $|b(t) + \cos\theta(t)|$ has been expanded in a Fourier-like series with time varying coefficients, i.e.,

$$\sum_{m=0}^{\infty} \varepsilon_m C(t,m) \cos m\theta(t) = |b(t) + \cos\theta(t)| \tag{B-17}$$

EVALUATION OF C(t,m)

$$C(t,m) = \frac{1}{2\pi j}\int_{\delta-j\infty}^{\delta+j\infty} C(t,\zeta)d\zeta \qquad \text{(B-18)}$$

where
$$C(t,\zeta) = \frac{I_m(\zeta)\left[e^{b\zeta} + (-1)^m e^{-b\zeta}\right]}{\zeta^2}$$

To evaluate the coefficients C(t,m), first consider the integral of C(t,ζ) around the contour shown in Fig. 36.

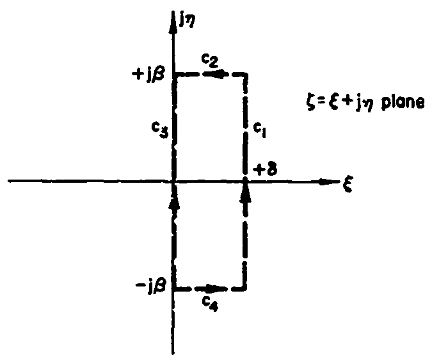

Figure 36. Contour of Integration for C(t,m) Coefficients

Defining K_1, K_2, K_3, K_4, as follows:

$$
\begin{aligned}
K_1 &= \int_{\delta-j\beta}^{\delta+j\beta} C(t,\zeta)d\zeta \quad , \quad \zeta = \delta + j\eta \\
K_2 &= \int_{\delta+j\beta}^{0+j\beta} C(t,\zeta)d\zeta \quad . \quad \zeta = \xi + j\beta \\
K_3 &= \int_{-j\beta}^{j\beta} C(t,\zeta)d\zeta \quad , \quad \zeta = j\eta \\
K_4 &= \int_{0-j\beta}^{\delta-j\beta} C(t,\zeta)d\zeta \quad , \quad \zeta = \xi - j\beta
\end{aligned}
\tag{B-19}
$$

Then as $\beta \rightarrow \infty, \frac{K_1}{2\pi j} \rightarrow C(t,m)$. Because

$$
I_m(z) = \frac{(\frac{z}{2})^m}{m!}\left[1 + \frac{(\frac{z}{2})^2}{m+1} + \frac{(\frac{z}{2})^4}{2(m+1)(m+2)} \cdots\right] \tag{B-20}
$$

the Bessel function $I_m(z)$ may be approximated by z^m for small values of z.

The discussion is initially restricted to the case $m > 2$, when the singularity of $C(t,\zeta)$ at the origin vanishes, and $C(t,\xi)$ becomes analytic inside, and on the contour of Fig. 36. It therefore follows from Cauchy's theorem that

$$
K_1 + K_2 - K_3 + K_4 = 0 \tag{B-21}
$$

The integrals $K_2 + K_4$ will now be considered. An asymptotic expansion for $I_m(z)$ for large values of $|z|$ is

$$
I_m(z) \doteq \frac{e^z}{\sqrt{2\pi z}}\left[1 - \frac{4m^2 - 1^2}{8z} + \frac{(4m^2 - 1^2)(4m^2 - 3^2)}{2!(8z)^2} \cdots\right] \tag{B-22}
$$

$$\therefore\ I_m(z) \doteq \frac{e^z}{\sqrt{2\pi z}} \tag{B-23}$$

Hence, for large values of β

$$K_2 = \frac{e^{j\beta}}{\sqrt{2\pi}}\int_{\delta}^{0}\frac{e^{\xi}\left[e^{b(\xi+j\beta)} + (-1)^m e^{-b(\xi+j\beta)}\right]d\xi}{(\xi + j\beta)^{5/2}} \tag{B-24}$$

$$|K_2| \le \frac{1}{\sqrt{2\pi}}\int_{\delta}^{0}\frac{e^{\xi}\left[e^{b(\xi+j\beta)} + (-1)^m e^{-b(\xi+j\beta)}\right]d\xi}{(\beta)^{5/2}} \tag{B-25}$$

$$|K_2| \le \frac{\left[\dfrac{e^{jb\beta}e^{(1+b)\xi}}{1+b} + \dfrac{(-1)^m e^{-jb\beta}(1-b)\xi}{1-b}\right]_{\delta}^{0}}{\sqrt{2\pi}\,\beta^{5/2}} \tag{B-26}$$

$$|K_2| \le \frac{e^{jb\beta}\left(\dfrac{1-e^{(1+b)\delta}}{1+b}\right) + (-1)^m e^{-jb\beta}\left(\dfrac{1-e^{(1-b)\delta}}{1-b}\right)}{\sqrt{2\pi}\,\beta^{5/2}} \rightarrow 0 \text{ as } \beta \rightarrow \infty \tag{B-27}$$

Thus, $K_2 \rightarrow 0$ and, similarly, $K_4 \rightarrow 0$ as $\beta \rightarrow \infty$. Hence,

$$K_1 = K_3 = \int_{\delta-j\infty}^{\delta+j\infty} C(t,\zeta)d\zeta \tag{B-28}$$

$$= \int_{-j\infty}^{j\infty} C(t,\zeta)d\zeta \tag{B-29}$$

$$K_3 = \int_{-j\infty}^{j\infty}\frac{I_m(\zeta)\left[e^{b\zeta} + (-1)^m e^{-b\zeta}\right]d\zeta}{\zeta^2} \tag{B-30}$$

Note that

$$C(t,m) = \frac{K_3}{2\pi_c} \tag{B-31}$$

On the imaginary axis

$$\zeta = j\eta \qquad d\zeta = j\,d\eta$$

$$K_3 = j^{m-1} \int_{-\infty}^{\infty} \frac{J_m(\eta)\left(e^{jb\eta} + (-1)^m e^{-jb\eta}\right)}{\eta^2}\, d\eta \tag{B-32}$$

because

$$I_m(j\eta) = j^m J_m(\eta) \tag{B-33}$$

for m even

$$K_3 = 2j^{m-1} \int_{-\infty}^{\infty} \frac{J_m(\eta)\cos b\eta}{\eta^2}\, d\eta = 4j^{m-1} \int_{0}^{\infty} \frac{J_m(\eta)\cos b\eta}{\eta^2}\, d\eta \tag{B-34}$$

for m odd

$$K_3 = 2j^{m} \int_{-\infty}^{\infty} \frac{J_m(\eta)\sin b\eta}{\eta^2}\, d\eta = 4j^{m} \int_{0}^{\infty} \frac{J_m(\eta)\sin b\eta}{\eta^2}\, d\eta \tag{B-35}$$

because

$$\begin{aligned} J_m(-\eta) &= (-1)^m J_m(\eta) \\ \sin(-b\eta) &= -\sin b\eta \end{aligned} \tag{B-36}$$

These integrals are Fourier cosine and Fourier sine transforms, respectively (Ref. 26). They are

$$Z = \frac{K_3}{4j^{m-1}} = \int_0^{\infty} \frac{J_m(\eta)\cos b\eta}{\eta^2}\, d\eta = \begin{cases} \dfrac{\dfrac{\cos(m-1)s}{m-1} + \dfrac{\cos(m+1)s}{m+1}}{2m} & b \leq 1,\ m \geq 2 \\[2ex] \dfrac{\sin\frac{m\pi}{2}}{2m}\left[-\dfrac{1}{(m-1)}\dfrac{1}{B^{m-1}} - \dfrac{1}{(m+1)\,B^{m+1}}\right] & b \geq 1,\ m = 0 \end{cases}$$

m even

(B-37)

$$W = \frac{K_3}{4J^2} = \int_0^{\infty} \frac{J_m(\eta)\sin b\eta}{\eta^2}\,d\eta = \begin{cases} \dfrac{c\sin ms}{m^2-1} - \dfrac{b\cos ms}{m(m^2-1)} & b \le 1,\ m \ge 3 \\[2ex] \dfrac{-\left(\cos\frac{m\pi}{2}\right)\left\{b + m\sqrt{b^2-1}\right\}}{m(m^2-1)B^m} & b \ge 1,\ m \ge 1 \end{cases} \qquad m \text{ odd} \tag{B-38}$$

where

$$s = \sin^{-1} b$$

$$c = \cos s = \sqrt{1-b^2}$$

$$B = b + \sqrt{b^2-1}$$

Applying L'Hospital's rule to the second terms of Z and W it can be shown that

$$Z\Big|_{m=0} = \frac{K_2}{4J^{m-1}}\Bigg|_{m=0} = \begin{cases} -(bs+c) & b \le 1 \\ \dfrac{-\pi b}{2} & b \ge 1 \end{cases} \tag{B-39}$$

$$W\Big|_{m=1} = \frac{K_2}{4J^m}\Bigg|_{m=1} = \begin{cases} \dfrac{1}{2}(bc+s) & b \le 1 \\ \dfrac{\pi}{4} & b \ge 1 \end{cases} \tag{B-40}$$

Because Z (for m even) and W (for m odd) are single-valued analytic functions of m for $m \ge 0$, the theory of analytic continuation (Ref. 16) permits the restriction $m > 2$ to be removed. The output signal can now be found by combining Eq B-15, B-31, B-37, and B-38.

$$y(t) = \begin{bmatrix} ae^{-\gamma t} + ke^{-\alpha t}\sin(\beta t + \psi) = e(t) & & b \ge 1 \\ \left\{\begin{array}{ll} \dfrac{2V}{\pi}\Big[\{bs+c\} + \{(bc+s)\sin(\beta t+\psi)\}\Big] & \\ -\dfrac{4V}{\pi}\displaystyle\sum_{m=2}^{\infty} Z\cos m(\beta t+\psi) & m \text{ even} \\ +\dfrac{4V}{\pi}\displaystyle\sum_{m=3}^{\infty} W\sin m(\beta t+\psi) & m \text{ odd} \end{array}\right\} & & b \le 1 \end{bmatrix} \tag{B-41}$$

where

$$b = \frac{a}{k} e^{-(\gamma-\alpha)t}$$

$$V = ke^{-\alpha t}$$

$$s = \sin^{-1} b$$

$$c = \cos s = \sqrt{1-b^2}$$

$$Z = \frac{mc \cos ms + b \sin ms}{m(m^2 - 1)}$$

$$W = \frac{mc \sin ms - b \cos ms}{m(m^2 - 1)}$$

NUMERICAL CHECK OF EXPRESSION FOR $|e(t)|$

Equation B-41 expresses $|e(t)|$ for a third-order system indicial response as a Fourier-like series with time-dependent coefficients. As will be shown, IAE and ITAE can be derived from this series by a straightforward procedure.

The usefulness of Eq B-41 depends upon the rapidity of the convergence of the series, i.e., how many harmonics must be included to obtain acceptable accuracy. One way of checking this point would be to evaluate IAE, and compare it with the published values in Ref. 37, or with values obtained by direct integration. However, as shown in Fig. 11 of Ref. 37, IAE is very insensitive to parameter changes. It would appear therefore that, with a small number of examples, a much more sensitive and thorough check can be obtained by computing $|e|$ at selected instants. Integration produces a smoothing effect so the number of harmonics required to represent IAE accurately will be less than the number required for $|e(t)|$.

Two responses are considered: the ITAE standard form and a lightly damped response. For the standard form, the parameters in Eq B-39 are

$$a = 1.2$$
$$k = 0.7245$$
$$\gamma = 0.707$$
$$\alpha = 0.52$$
$$\beta = 1.07$$
$$\psi = -0.279 \text{ rad or } -16 \text{ deg}$$

and therefore e(t), Eq B-9, becomes

$$e(t) = 1.2e^{-0.707t} + 0.7245e^{-0.32t} \sin (1.07t - 0.279) \qquad \text{(B-42)}$$

For the lightly damped system the following parameter values were selected:

$$a = 0.4264$$
$$k = 0.798$$
$$\gamma = 1.142$$
$$\alpha = 0.0542$$
$$\beta = 0.935$$
$$\psi = \pi/2 - 0.7611$$

and therefore e(t), Eq B-9, becomes

$$e(t) = 0.4264e^{-1.142t} + 0.798e^{-0.0542t} \sin (0.935t + \pi/2 - 0.7611) \qquad \text{(B-43)}$$

The standard form |e(t)| was examined at t = 3.20, 4.55, 6.13, and 6.41 nondimensional sec, these values being chosen for convenience of calculation. The lightly damped response was examined at t = 0.4105 and 3.66 normalized sec. The results are illustrated in Fig. 37 through 42, which show that the use of only the first five harmonics gives values within 7 percent of the true |e(t)|. By taking 6 or 7 harmonics accuracy of within 2 or 3 percent can be obtained.

EVALUATION OF IAE

The integral of the absolute error can be found from Eq B-41 for y (t)

$$IAE = \int_0^{\infty} y(t)dt = H_0 + \sum_{m=0}^{\infty} U_m \qquad \text{(B-44)}$$

where

$$H_0 = \int_0^{T_1} e(t)dt$$

$$\sum_{m=0}^{\infty} U_m = \int_{T_1}^{\infty} |e(t)|dt$$

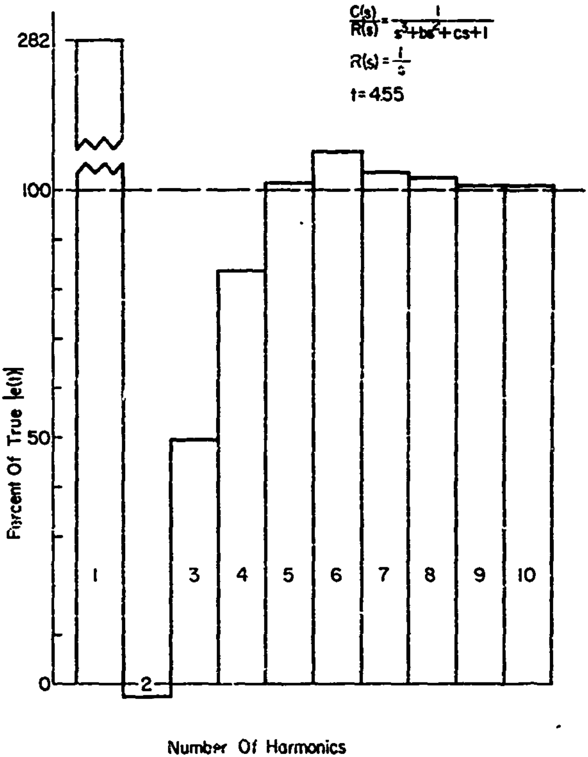

Figure 37. Calculated Value of $|e(t)|$ for Third-Order Optimal ITAE System vs Number of Harmonics at $t = 4.55$ Normalized Sec

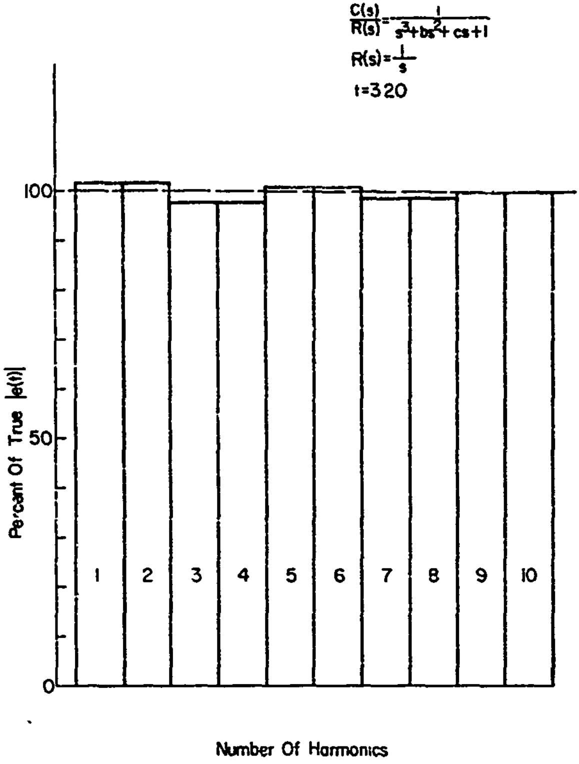

Figure 38. Calculated Value of $|e(t)|$ for Third-Order Optimal ITAE System vs Number of Harmonics at $t = 3.20$ Normalized Sec

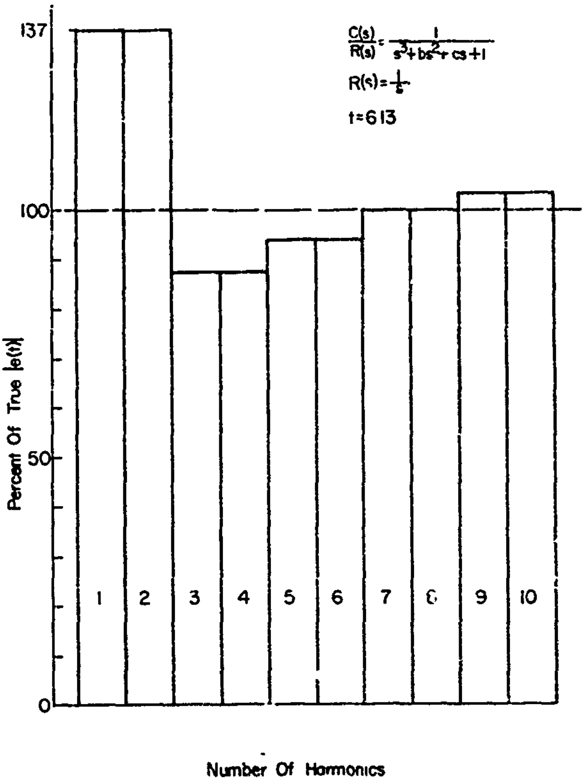

Figure 39. Calculated Value of $|e(t)|$ for Third-Order Optimal ITAE System vs Number of Harmonics at t = 6.13 Normalized Sec

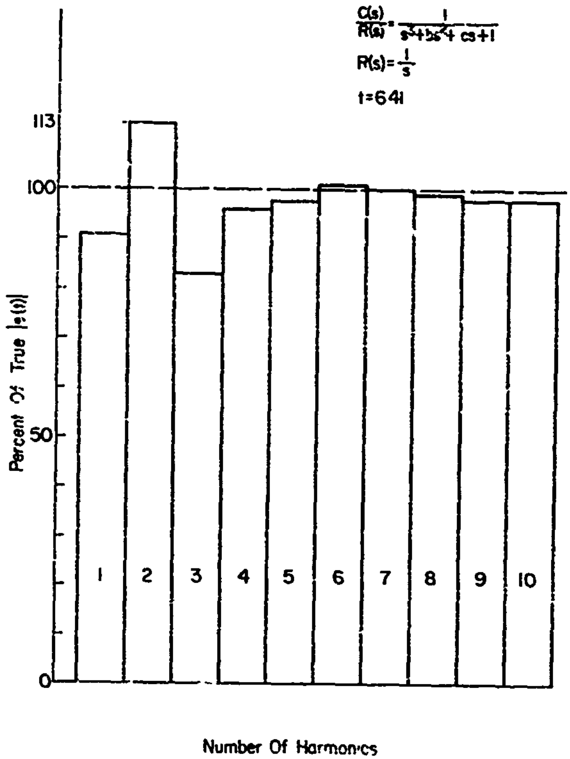

Figure 40. Calculated Value of $|e(t)|$ for Third-Order Optimal ITAE System vs Number of Harmonics at t = 6.41 Normalized Sec

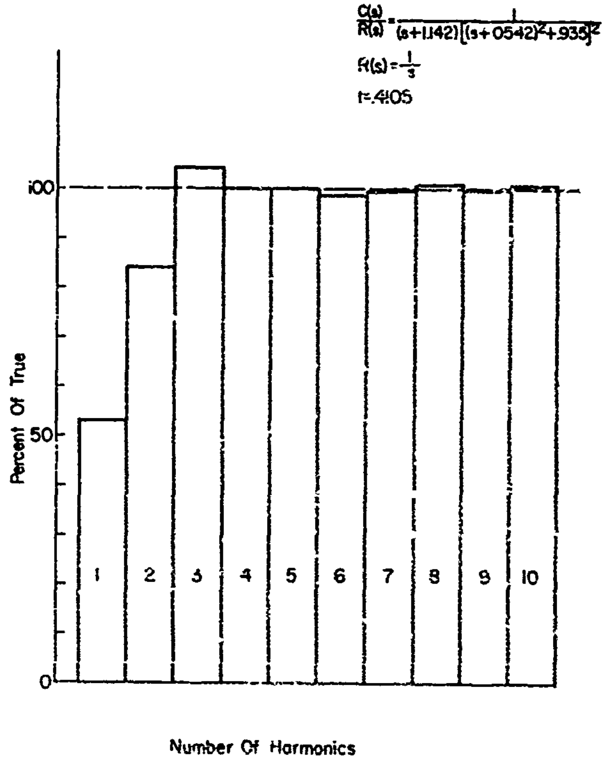

Figure 41. Calculated Value of $|c(t)|$ for Lightly Damped System vs Number of Harmonics at t = 0.4105 Normalized Sec

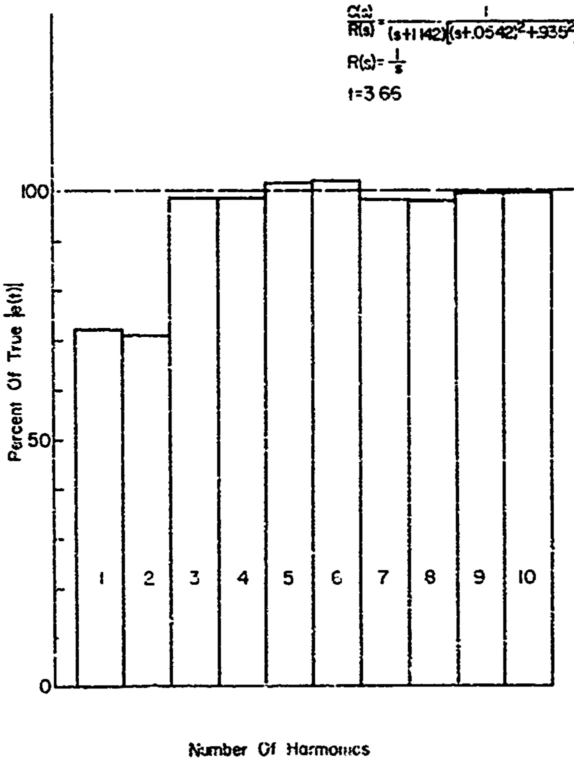

Figure 42. Calculated Value of $|e(t)|$ for Lightly Damped System vs Number of Harmonics at t = 3.66 Normalized Sec

$$b(T_1) = 1$$

$$\therefore T_1 = \frac{\ln \frac{a}{k}}{\gamma - \alpha}$$

The various terms of H_0 can be evaluated in a straightforward manner.

$$U_0 = \frac{a(1 - e^{-\gamma T_1})}{\gamma} + \gamma^{1/2} k \left[\sin(\phi + \gamma) - e^{-\alpha T_1} \sin(\psi + \gamma)\right] \tag{B-45}$$

where

$$\gamma = \tan^{-1} \frac{\beta}{\alpha}$$
$$\phi = \beta T_1 + \psi$$

The procedure for integrating the U_m forms is more complicated, and will now be described in detail.

Evaluation of U_0

From Eq B 41 and Eq B 44,

$$U_0 = \frac{2k}{\pi} \int_{T_1}^{\infty} (bs + c)e^{-\alpha t}dt \tag{B-46}$$

let

$$t = \tau + T_1$$

$$p = \alpha$$

Note that α is not replaced by p in b, s, or c. The resulting expressions can be likened to Laplace transforms. Other advantages will accrue when ITAE is evaluated.

$$\therefore U_0 = A_0 \int_0^{\infty} (b_1 s_1 + c_1)e^{-p\tau}d\tau \tag{B-47}$$

where

$$A_0 = \frac{2ke^{-\alpha T_1}}{\pi}$$
$$b_1 = e^{-\delta t}$$
$$\delta = \gamma - \alpha$$

$$c_1 = \sqrt{1 - b_1^2}$$

$$s_1 = \sin^{-1} b_1$$

$$\frac{U_0}{A_0} = \int_0^{\infty} \sin^{-1} b_1 e^{-(p+g)\tau} d\tau + \int_0^{\infty} (1 - b_1^2)^{1/2} e^{-p\tau} d\tau \tag{B-48}$$

integrating the first term by parts

$$\frac{U_0}{A_0} = \left[\frac{(\sin^{-1} b_1) e^{-(p+g)\tau}}{-(p + g)}\right]_0^{\infty} + \frac{-g}{p + g} \int_0^{\infty} \frac{e^{-(p+2g)\tau} d\tau}{\sqrt{1 - b_1^2}} + \int_0^{\infty} (1 - b_1^2)^{1/2} e^{-p\tau} d\tau \tag{B-49}$$

using

$$\int_0^{\infty} (1 - e^{-2g\tau})^q e^{-p\tau} d\tau = \frac{B(\frac{p}{2g}, q + 1)}{2g} \text{ (Ref. 26)} \tag{B-50}$$

Eq B-49 becomes

$$\frac{U_0}{A_0} = \frac{\pi - B(\frac{p + 2g}{2g}, 1/2)}{2(p + g)} + \frac{B(\frac{p}{2g}, 3/2)}{2g} \tag{B-51}$$

where $\quad B(z, g_0) = \dfrac{\Gamma(z)\Gamma(g_0)}{\Gamma(z + g_0)}$ is the Beta function

and where $\quad z = x + jy$ is a general complex variable

<u>Evaluation of U_1</u>

From Eq B-41 and B-44

$$U_1 = \frac{2k}{\pi} \int_{T_1}^{\infty} (bc + s) \sin (\beta t + \psi) e^{-\alpha t} dt \tag{B-52}$$

let

$$t = \tau + T_1$$

$$p = \alpha$$

$$\frac{U_1}{A_0} = \int_0^{\infty} (b_1c_1 + s_1) \sin (\beta\tau + \psi + \beta T_1) e^{-p\tau} d\tau \qquad \text{(B-53)}$$

let

$$w_1(\tau) = b_1c_1 + s_1$$

$$W_1(p) = \mathcal{L}\left[w_1(\tau)\right] = \int_0^{\infty} (b_1c_1 + s_1) e^{-p\tau} d\tau$$

$$\therefore \frac{U_1}{A_0} = \cos \phi \left(\frac{W_1(p - j\beta) - W_1(p + j\beta)}{2j}\right) + \sin \phi \left(\frac{W_1(p - j\beta) + W_1(p + j\beta)}{2}\right) \qquad \text{(B-54)}$$

where

$$\phi = \psi + \beta T_1$$

The expression for $W_1(p)$ is similar to that for U_0, the difference being the replacement of p by p - g in one term, and by p + g in the other term.

$$W_1(p) = \frac{\pi - B(\frac{p + g}{2g}, 1/2)}{2p} + \frac{B(\frac{p + g}{2g}, 3/2)}{2g} \qquad \text{(B-55)}$$

The Beta function with one complex argument can be written as either an infinite series or an infinite product (Ref. 52). The actual calculation of IAE for specific parameter values is simpler with the infinite product, but the infinite series allows a simple derivation of ITAE, as will be demonstrated.

The Beta function with one complex argument can be written

$$g_0 B(z, g_0) = \sum_{n=0}^{\infty} \frac{(-1)^n g_0(g_0 - 1)(g_0 - 2) \cdots (g_0 - n)}{(n!)} \left(\frac{1}{z + n}\right) \qquad \text{(B-56)}$$

with g_0 real and positive and z complex

$$\therefore \frac{B(x - jy, g_0) - B(x + jy, g_0)}{2j} = \sum_{n=0}^{\infty}\left(\frac{(-1)^n g_0(g_0 - 1)(g_0 - 2)\cdots(g_0 - n)}{g_0(n!)}\right)\cdot \left(\frac{y}{(x + n)^2 + y^2}\right)$$

$$\frac{B(x - jy, g_0) + B(x + jy, g_0)}{2} = \sum_{n=0}^{\infty}\left(\frac{(-1)^n g_0(g_0 - 1)(g_0 - 2)\cdots(g_0 - n)}{g_0(n!)}\right)\cdot \left(\frac{x + n}{(x + n)^2 + y^2}\right) \tag{B-57}$$

Note that the first coefficient of each series is unity, not g_0. On combining Eq B-54, B-55, and B-57 U_1 becomes

$$\frac{U_1}{A_0} = \left\{\begin{aligned} &\frac{\pi}{2}\frac{(\beta\cos\phi + p\sin\phi)}{p^2 + \beta^2} \\ &+ \sum_{n=0}^{\infty}\frac{(-1)^n a_1(a_1 - 1)(a_1 - 2)\cdots(a_1 - n)}{a_1 n!}\left[\frac{\beta\cos\phi + (p + g + 2gn)\sin\phi}{(p + g + 2gn)^2 + \beta^2}\right] \\ &- \sum_{n=0}^{\infty}\left(\frac{(-1)^n a_0(a_0 - 1)(a_0 - 2)\cdots(a_0 - n)}{a_0 n!}\right)\cdot \\ &\left(\frac{\beta\left[\frac{p}{2g} + \frac{p + g}{2g} + n\right]\cos\phi + \left[p\left(\frac{p + g}{2g} + n\right) - \frac{\beta^2}{2g}\right]\sin\phi}{2\left[\left(\frac{p + g}{2g} + n\right)^2 + \frac{\beta^2}{4g^2}\right](p^2 + \beta^2)}\right) \end{aligned}\right. \tag{B-58}$$

where

$$a_0 = 1/2$$

$$a_1 = 3/2$$

$$\phi = \psi + \beta T_1$$

The Beta function with one complex argument can also be written (Ref. 58, page 2)

$$B(x + jy, g_0) = \frac{\Gamma(g_0)\Gamma(x + jy)}{\Gamma(x + g_0 + jy)} = \left(\frac{\Gamma(g_0)\Gamma(x)}{\Gamma(x + g_0)}\right) \cdot \left(\frac{e^{-jcy}\left(\frac{x}{x + jy}\right)\prod_{n=1}^{\infty}\left[\frac{e^{jy/n}}{1 + \frac{jy}{x + n}}\right]}{e^{-jcy}\frac{(x + g_0)}{(x + g_0 + jy)}\prod_{n=1}^{\infty}\left[\frac{e^{jy/n}}{1 + \frac{jy}{x + g_0 + n}}\right]}\right) \tag{B-59}$$

where

$$c = \text{Eulers constant} = 0.577 \ldots$$

$$\therefore B(x + jy, g_0) = B(x, g_0)\prod_{n=0}^{\infty}\left[\frac{1 + \frac{jy}{x + g_0 + n}}{1 + \frac{jy}{x + n}}\right] \tag{B-60}$$

When Eq B-55 and B-60 are substituted into Eq B-54, $\frac{U_1}{A_0}$ becomes

$$\frac{U_1}{A_0} = \frac{x(p \cos \phi + p \sin \phi)}{2(p^2 + \beta^2)} + \frac{k_1}{2g}\sin(\phi + \Delta_1) - \frac{k_2}{2}\sin(\phi + \Delta_2) \tag{B-61}$$

where

$$k_1 = B\left(\frac{p + g}{2g}, a_1\right)\left|\underbrace{\prod_{n=0}^{\infty}\left(\frac{1 - \frac{j\beta}{p + g + 2g(a_1 + n)}}{1 - \frac{j\beta}{p + g + 2gn}}\right)}_{M_1}\right|$$

$$\Delta_1 = \angle M_1 = \sum_{n=0}^{\infty}\left(\tan^{-1}\frac{\beta}{p + g + 2gn} - \tan^{-1}\frac{\beta}{p + g + 2g(a_1 + n)}\right)$$

$$k_2 = B\left(\frac{p+g}{2g}, a_0\right) \underbrace{\left| \frac{1}{p - j\beta} \prod_{n=0}^{\infty} \left(\frac{1 - \frac{j\beta}{p + g + 2g(a_0 + n)}}{1 - \frac{j\beta}{p + g + 2gn}} \right) \right|}_{M_2}$$

$$\Lambda_2 = \measuredangle M_2 = \tan^{-1}\frac{\beta}{p} + \sum_{n=0}^{\infty}\left(\tan^{-1}\frac{\beta}{p + g + 2gn} - \tan^{-1}\frac{\beta}{p + g + 2g(a_0 + n)}\right)$$

$$a_1 = 3/2$$

$$a_0 = 1/2$$

The evaluation of k_1, k_2, Δ, and Δ_2 can be performed by operations on a Bode diagram because M_1 and M_2 consist of an alternating set of poles and zeros if $j\beta$ is regarded as the Laplace variable s. Only those terms with breakpoints less than a few times β need be included, because each pair of poles and zeros are quite close together.

Evaluation of U_2

From Eq B-41 and B-44

$$U_2 = \frac{-4k}{\pi} \int_{T_1}^{\infty} Z(t) \cos 2(\beta t + \psi) e^{-\alpha t} dt \tag{B-62}$$

Let

$$t = \tau + T_1$$

$$p = \alpha$$

$$\therefore U_2 = -2A_0 \int_0^{\infty} \left(\frac{2c_1 \cos 2s_1 + b_1 \sin 2s_1}{6} \right) \cos 2(\beta\tau + \psi + \beta T_1) e^{-p\tau} d\tau \tag{B-63}$$

$$\therefore \frac{-3U_2}{2A_0} = \int_0^{\infty} (1 - b_1^2)^{3/2} \cos 2(\beta\tau + \psi + \beta T_1) e^{-p\tau} d\tau \tag{B-64}$$

let
$$w_2(\tau) = (1 - b_1^2)^{3/2}$$

$$\therefore W_2(p) = \mathcal{L}\left[w_2(\tau)\right] = \frac{b\left(\frac{p}{2g}, a_2\right)}{2g} \qquad \text{(B-65)}$$

where
$$a_2 = 5/2$$

$$\frac{-3U_2}{2A_0} = \left(\frac{W_2(p - j2\beta) + W(p + j2\beta)}{2}\right)\cos 2\phi - \left(\frac{W_2(p - j2\beta) - W_2(p + j2\beta)}{2j}\right)\sin 2\phi \qquad \text{(B-66)}$$

Combining Eq B-65 and B-66, and using Eq B-57, U_2 becomes

$$\frac{-3U_2}{2A_0} = \sum_{n=0}^{\infty} \frac{(-1)^n a_2(a_2 - 1)(a_2 - 2) \cdots (a_2 - n)}{a_2 n!} \cdot$$

$$\left(\frac{(p + 2gn)\cos 2\phi - 2\beta \sin 2\phi}{(p + 2gn)^2 + 4\beta^2}\right) \qquad \text{(B-67)}$$

Evaluation of U_m, $M \geq 3$, m odd,

From Eq B-41 and B-44

$$U_m = \frac{4k}{\pi} \sum_{m=3}^{\infty} \int_{T_1}^{\infty} \left(\frac{a \sin ms - b \cos ms}{m(m^2 - 1)}\right) \sin m(\beta t + \psi) e^{-\alpha t} dt \qquad \text{(B-68)}$$

let
$$t = \tau + T_1$$

$$p = \alpha$$

$$\frac{m(m^2-1)U_m}{2A_0} = \sum_{\substack{m=3\\ m\ \text{odd}}}^{\infty} \int_0^{\infty} v_m(\tau)\,\sin m(\beta\tau + \psi + \beta\tau_1)\,e^{-p\tau}\,d\tau \tag{B-69}$$

where

$$v_m(\tau) = mc_1 \sin ms_1 - b_1 \cos ms_1$$

$$W_m(p) = \mathcal{L}\left[v_m(\tau)\right]$$

$$\frac{m(m^2-1)U_m}{2A_0} = \sum_{\substack{m=3\\ m\ \text{odd}}}^{\infty} \left\{ \begin{aligned} &\frac{W_m(p - jm\beta) - W(p + jm\beta)}{2j}\cos\phi \\ &\frac{W_m(p - jm\beta) + W_m(p + jm\beta)}{2}\sin\phi \end{aligned} \right\} \tag{B-70}$$

$\cos ms_1$ and $\sin ms_1$ can be expanded by the following formulae (Ref. 25):

$$\underset{\substack{m \geq 3\\ m\ \text{odd}}}{\cos ms_1} = \sum_{r=0}^{\frac{m-1}{2}} \frac{(-1)^r 2^{m-2r-1} m(m-r-1)!}{r!(m-2r)!} c_1^{m-2r}$$

$$\underset{\substack{m \geq 3\\ m\ \text{odd}}}{\sin ms_1} = (-1)^{\frac{m-1}{2}} \sum_{r=0}^{\frac{m-1}{2}} \frac{(-1)^r 2^{m-2r-1} m(m-r-1)!}{r!(m-2r)!} b_1^{m-2r} \tag{B-71}$$

With these expressions for $\cos ms_1$ and $\sin ms_1$ $W_m(p)$ becomes

$$W_m(p) = \sum_{r=0}^{\frac{m-1}{2}} \frac{(-1)^r 2^{m-2r-1} m(m-r-1)!}{r!(m-2r)!}\,\omega_m(p) \tag{B-72}$$

where

$$\omega_m(p) = \int_0^{\infty} \left((-1)^{\frac{m-1}{2}} mc_1 b_1^{m-2r} - b_1 c_1^{m-2r} \right) e^{-p\tau}\,d\tau$$

$$\Phi_m(p) = \frac{mB\left(\frac{p + (m - 2r)g}{2g}, 3/2\right) - B\left(\frac{p + g}{2g}, \frac{m - 2r + 2}{2}\right)}{2g}$$

U_m is obtained by combining Eq B-72, B-57 and B-70. The resulting expression is rather complicated, and is not given here.

Evaluation of U_m $m \geq 4$, m even

From Eq B-4 and B-44

$$U_m = \frac{-4k}{\pi} \sum_{\substack{m=4 \\ m \text{ even}}}^{\infty} \int_{T_1}^{\infty} \left(\frac{mc \cos ms + b \sin ms}{m(m^2 - 1)}\right) \cos m(\beta t + \psi) e^{-\alpha t} dt \qquad \text{(B-73)}$$

let
$$t = \tau + T_1$$

$$p = \alpha$$

$$\frac{-m(m^2 - 1)}{2A_0} U_m = \sum_{\substack{m=4 \\ m \text{ even}}}^{\infty} \int_0^{\infty} w_m(\tau) \cos m(\beta\tau + \phi) e^{-p\tau} d\tau \qquad \text{(B-74)}$$

where

$$w_m(\tau) = mc_1 \cos ms_1 + b_1 \sin ms_1$$

$$\phi = \psi + \beta T_1$$

$$\frac{-m(m^2 - 1)U_m}{2A_0} = \sum_{\substack{m=4 \\ m \text{ even}}}^{\infty} \left\{ \begin{array}{l} \left\{\dfrac{W_m(p - jm\beta) + W_m(p + jm\beta)}{2}\right\} \cos m\phi \\ - \left\{\dfrac{W_m(p - jm\beta) - W_m(p + jm\beta)}{2j}\right\} \sin m\phi \end{array} \right\} \qquad \text{(B-75)}$$

where $$W_m(p) = \mathcal{L}\left[w_m(\tau)\right]$$

$\cos ms_1$ and $\sin ms_1$ can be expanded by the following formulae.

$$\cos ms_1 = \sum_{r=0}^{\infty} \frac{(-1)^r 2^{m-2r-1} m(m-r-1)!\, c_1^{m-2r}}{r!(m-2r)!} \qquad m \geq 3 \tag{B-76}$$

$$\sin ms_1 = (-1)^{\frac{m}{2}+1} c_1 \sum_{r=0}^{\infty} \frac{(-1)^r 2^{m-2r-1}(m-r-1)!\, b_1^{m-2r-1}}{r!(m-2r-1)!} \qquad m \geq 2,\ m \text{ even} \tag{B-77}$$

Because $W_m(p)$ is the Laplace tranrform of $w_m(\tau)$

$$W_m(p) = \int_0^{\infty} (mc_1 \cos ms_1 \quad b_1 \sin ms_1) e^{-p\tau} d\tau \qquad m \text{ even} \tag{B-78}$$

$$W_m(p) = \sum_{r=0}^{m/2} \frac{(-1)^r 2^{m-2r-1}(m-r-1)!}{r!(m-2r)!} \omega_m(p) \qquad m \text{ even} \tag{B-79}$$

where

$$\omega_m(p) = \int_0^{\infty}\left(m^2(1-b_1^2)^{\frac{m-2r+1}{2}} e^{-p\tau} + (-1)^{m/2+1}(m-2r)\sqrt{1-b_1^2}\, e^{-[p+g(m-2r)]\tau}\right)d\tau$$

$$\omega_m(p) = \frac{m^2 B\left(\frac{p}{2g}, \frac{m-2r+3}{2}\right) + (-1)^{m/2+1}(m-2r) B\left(\frac{p+g(m-r)}{2g}, 3/2\right)}{2g}$$

By combining Eq B-79, B-57, and B-75 the expression for U_m can be found. Again the expression is complicated, and will not be given here.

CALCULATION OF IAE FOR OPTIMUM THIRD-ORDER SYSTEM

The parameters required for the calculation of IAE for the third-order system are given just prior to Eq B-42. The values of the first few components of IAE are

$$
\begin{aligned}
I &= 1.98 \\
U_0 &= 0.285 \\
\underline{U_1} &= \underline{-0.109} \\
\text{Total} &= 2.156
\end{aligned}
\tag{B-80}
$$

The value of IAE scaled off of Fig. 11 of Ref. 37 is approximately 2.05, and the calculated value of ITAE is therefore within approximately 5 or 6 percent of the experimentally obtained value. The inclusion of more components of IAE would improve the accuracy.

The infinite product representation (Eq B-61) for the Beta function was used for the numerical calculation of IAE. Only the first four factors were used, and the accuracy was judged to be acceptable.

EVALUATION OF ITAE

ITAE is defined by

$$\text{ITAE} = \int_0^{\infty} t|e(t)|dt \tag{B-81}$$

Because the expression used for $|e|$ is in two parts

$$\text{ITAE} = \int_0^{T_1} te\,dt + \int_{T_1}^{\infty} t|e|dt \tag{B-82}$$

It is convenient to define

$$H_1 = \int_0^{T_1} te\,dt\ , \quad \sum_{m=0}^{\infty} N_m = \int_{T_1}^{\infty} t|e|dt \tag{B-83}$$

i.e., each H_m corresponds to the m'th component of the contribution of the second form in Eq B-82 to ITAE.

The expressions for H_1 and the H_m terms are rather involved and lengthy. Numerical checks will not be attempted, but H_1 and N_0 will be evaluated in literal form because the calculation of these two expressions is the most difficult part of evaluating ITAE.

From Eq B-81 and B-83,

$$H_1 = \int_0^{T_1} tc\,dt = a\int_0^{T_1} te^{-\gamma t}dt + k\int_0^{T_1} t\sin(\beta t + \psi)e^{-\alpha t}dt \tag{B-84}$$

The first of these integrals is quite simple, but the second is rather involved. H_1 can be rewritten as (integrating the first term)

$$H_1 = \begin{cases} \dfrac{a}{\gamma^2}\left\{1 - e^{-\gamma T_1}(1 - \gamma T_1)\right\} + k\displaystyle\int_0^{\infty} t\,(\sin\beta t\cos\psi + \cos\beta t\sin\psi)e^{-\alpha t}dt \\ \quad - k\displaystyle\int_{T_1}^{\infty} t\sin(\beta t + \psi)e^{-\alpha t}dt \end{cases} \tag{B-85}$$

The second term of Eq B-85 is the Laplace transform of $t\sin\beta t$ and $t\cos\beta t$ with α replacing the Laplace variable s. The third term can be easily evaluated by making a change of variable $t = \tau + T_1$. Upon integrating the second term, H_1 becomes

$$H_1 = \frac{a}{\gamma^2}\left(1 - e^{-\gamma T_1}(1 - \gamma T_1)\right) + k\gamma\sin(\psi + \omega_3)$$

$$- ke^{-\alpha T_1}\int_0^{\infty}(\tau + T_1)(\sin\beta\tau\cos\psi + \cos\beta\tau\sin\psi)e^{-\alpha\tau}d\tau \tag{B-86}$$

Finally,

$$N_1 = \frac{a}{\gamma^2}\left\{1 - e^{-\gamma T_1}\left(1 - \gamma T_1\right)\right\}$$

$$+ k\left[\gamma \sin(\phi + \Delta_3) - e^{-\alpha T_1}\left\{\gamma \sin(\phi + \Delta_3) + \gamma^{1/2} T_1 \sin(\phi + \Delta_4)\right\}\right] \qquad \text{(B-87)}$$

where

$$\Delta_3 = \tan^{-1}\frac{2\beta\alpha}{\alpha^2 - \beta^2}$$

$$\Delta_4 = \tan^{-1}\frac{\beta}{\alpha}$$

<u>Evaluation of N_0</u>

From Eq B-41 and B-83,

$$N_0 = \frac{2k}{\pi}\int_{T_1}^{\infty} t(bt + c)e^{-\alpha t}dt \qquad \text{(B-88)}$$

let

$$t = \tau + T_1$$

$$p = \alpha$$

$$N_0 = A_0\int_0^{\infty}(\tau + T_1)(b_1\tau + c_1)e^{-p\tau}d\tau \qquad \text{(B-89)}$$

The terms in this expression are similar to those for U_0 except for the factor $(\tau + T_1)$ in the integrand. Since the expression can be likened to a Laplace transform

$$N_0 = -\frac{\partial U_0}{\partial p} + T_1 U_0 \qquad \text{(B-90)}$$

U_0 is given by Eq B-51, but the derivative with respect to p offers some difficulty. Representation of the Beta function by an infinite series (Eq B-56) simplifies the differentiation considerably because p occurs only in the term $1/(z + n)$.

The derivative with respect to p of $1/z + n$ is merely $-1/(z + n)^2$ with a multiplicative constant. Accordingly,

$$\frac{K_0}{A_0} = \frac{\pi - B(\frac{p}{2g} + 1,\ 1/2)}{2(p + g)^2} + \frac{1}{2(p + g)} \sum_{n=0}^{\infty} \frac{(-1)^{n+1} a_0(a_0 - 1)(a_0 - 2) \cdots (a_0 - n)}{a_0 2g(n!)(\frac{p}{2g} + 1 + n)^2}$$

$$+ \frac{1}{4g^2} \sum_{n=0}^{\infty} \frac{(-1)^r a_1(a_1 - 1)(a_1 - 2) \cdots (a_1 - n)}{a_1(r!)(\frac{p}{2g} + n)^2} + T_1 \frac{U_0}{A_0} \qquad \text{(B-91)}$$

The expressions for K_m, $m \geq 1$, will be similar to Eq B-89 and B-90

$$K_m = -\frac{\partial U_m}{\partial p} + T_1 U_m \qquad \text{(B-92)}$$

The desirability of the series representation for the Beta function is evident. Note that the terms of K_m should converge more rapidly than those of U_m because there is generally an additional factor of the form $1/(z + n)$ in each summation with respect to n.

APPENDIX C

DETAILS OF THE EXAMPLE FLIGHT CONTROL SYSTEM USED TO DEMONSTRATE THE EQUIVALENT SYSTEM CONCEPT

DETAILS OF THE EXAMPLE FLIGHT CONTROL SYSTEM USED TO DEMONSTRATE THE EQUIVALENT SYSTEM CONCEPT

Equation 1 of Chapter I in the main body of the report expresses the open-loop transfer function relating pitch attitude to elevator deflection for the fighter airplane detailed in Table III-1 of Ref. 2. The altitude is 20,000 ft, the weight is 30,000 lb, and true airspeed is 660 ft/sec (Mach No. = 0.638). The airframe transfer function is as quoted in Ref. 2,

$$\frac{\theta(s)}{\delta_e(s)} = \frac{4.85\left(\frac{s}{1.372} + 1\right)\left(\frac{s}{0.0098} + 1\right)}{\left[\frac{s^2}{(0.0650)^2} + \frac{2(0.0714)}{0.0650}s + 1\right]\left[\frac{s^2}{(4.27)^2} + \frac{2 \times 0.493}{4.27}s + 1\right]} \tag{C-1}$$

The servomotor plus amplifier transfer function is estimated as

$$\frac{\delta_e(s)}{\theta(s)} = \frac{K_m}{\frac{s^2}{(50)^2} + \frac{2(0.7)s}{50} + 1} \tag{C-2}$$

The equalization is described by $K_1\left(\frac{s}{2.4} + 1\right)$. Combining this with the product of C-1 and C-2 yields the open-loop transfer function of the complete system.